Chapter 2

Distances and Brightness

As we mentioned in the previous chapter, the objects in the sky are so distant from Earth (or the size of the Earth in comparison to the dimensions of the sky are so small) that we lose the notion of depth, and all objects seem incrusted in a dome, the celestial sphere. That begs the question of how do we measure the distances from objects to the Earth. How far is the Moon? How far is the Sun? How far are the planets? How far are the stars and the other objects in the night sky? Answering this question is by no means trivial and has motivated astronomers throghout millenia. The effort led to several methods for estimating distances, leading to what we call the *distance ladder*. In this chapter we will study the first step in the ladder, which is the *parallax*.

2.1 Parallax

Parallax is the shift in position of a fixed object relative to a distant background when viewed from different points (Fig. 2.1). If the observer is in motion, it is the apparent motion of a fixed object against a distant background because of the motion of the observer.

Hipparchus could determine the distance to the Moon to be about 59 Earth radii by using the concept of parallax. He knew of a total eclipse of the Moon in the Hellespont (modern day strait of Dardanelles), and got word that the same eclipse was only partial in Alexandria, with the Moon covering only 4/5 of the Sun (Fig. 2.2). The disk of the Sun has a diameter of 30 minutes, so in Alexandria the limb of the Moon was $30/5 = 6' = 0.1^\circ$ away from its position seen from the Hellespont. Hipparchus knew that Alexandria and the Hellespont were 9° of latitude away and almost exactly North-South (so same longitude). From plane trigonometry, the baseline is $d = R\theta$, where d is the distance between the cities, R is the distance to the moon, and θ the parallax angle. So,

$$R = \frac{d}{\theta} \approx \frac{R_\oplus \times 9^\circ}{0.1^\circ} \approx 90 R_\oplus \tag{2.1}$$

The actual value is 60.32 $R_\oplus$. So, Hipparchus got very close, to within a factor 2.

Aristarchus of Samus had also meaured the distance to the Moon, using an eclipse. He reasoned that the shadow of the Earth was equal to its diameter, and measured 3 hours for the center of the moon to move in and out of the eclipse. That means that if in

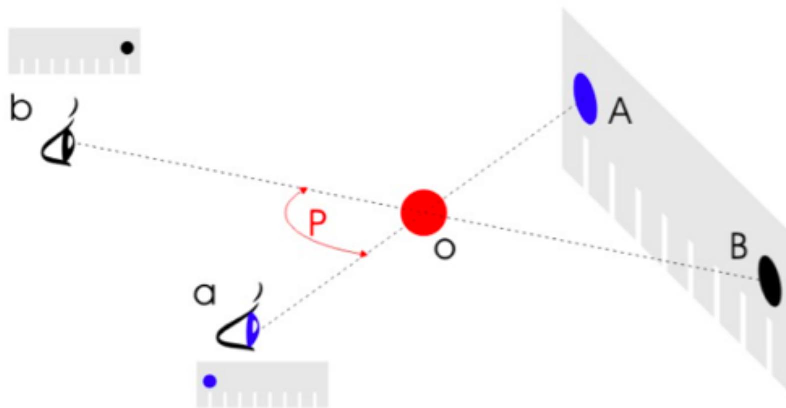

Figure 2.1: Parallax is the shift in position of a fixed object relative to a distant background when viewed from different points.

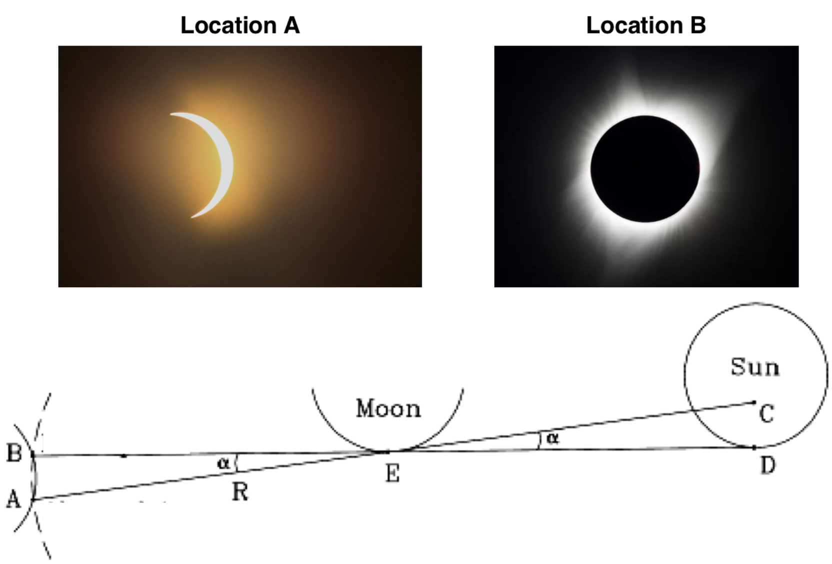

Figure 2.2: Hipparchus determined the distance to the Moon by measuring the lunar parallax during a solar eclipse, from two different points on Earth. He found a distance close to the actual value of 60 Earth radii.

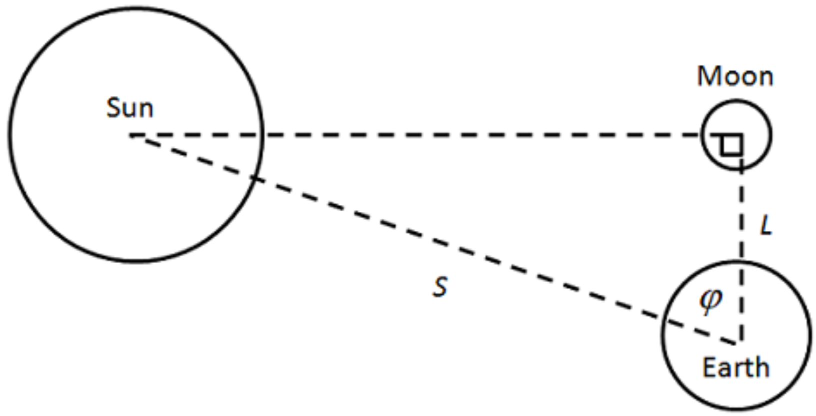

Figure 2.3: Aristarchus' method of determining the distance to the Sun. At the moment of 1st or 3rd quarter, the angle formed by the Sun, the Moon, and the Earth, centered on the Moon, is a right angle. Measuring the angle φ and knowing the distance L to the Moon would yield the distance S to the Sun. Aristarchus measured $\varphi = 87°$, and thus $S = 20L$. This value is a severe underestimate. The angle φ is actually $89°51'$, i.e., it deviates from $90°$ by mere $9'$, not $3°$, leading to $S \approx 400L$. Aristarchus' $3°$ was the precision of the observation.

a month T the Moon goes the whole circumference of the orbit, and it took t =3 hours to move $2R_\oplus$, he could tell the size of the orbit

$$\frac{T}{t} = \frac{2\pi d}{2R_\oplus}. \tag{2.2}$$

Given the values

$$\frac{d}{R_\oplus} \approx \frac{28\,\mathrm{day} \times 24\,\mathrm{h/day}}{\pi \times 3\,\mathrm{h}} \approx 71; \tag{2.3}$$

again a remarkably close result.

2.1.1 Distance to the Sun

Aristarchus of Samus also tried to estimate the distance to the Sun. For that, he used the fact that at a first or third quarter, the Earth, the Moon, and the Sun form the vertices of a right triangle (see Fig. 2.3). The angle φ, measured in the sky, between the Sun and the Moon, would thus provide the distance S between the Earth and the Sun, if the Earth-Moon distance L is known. From Fig. 2.3, we see that

$$\cos\varphi = \frac{L}{S} \tag{2.4}$$

Aristarchus measured $\varphi = 87°$, thus yielding

$$S = \frac{L}{\cos 87°} \approx 20L \tag{2.5}$$

The true value is actually closer to $400L$, so 20 times larger. The problem is ill-posed, with φ very close to 90°. It deviates from 90° by barely 9′, far from the positional accuracy of the ancients. Aristarchus' measurement that φ deviated from 90° by 3° was simply the precision of his observation and thus, in modern parlance, a lower limit to the distance to the Sun.

2.1.2 Geocentric Parallax

Consider Fig. 2.4. For a hypothetical observer at the center of the Earth, a star is at zenith distance z. For an observer at O, the zenith angle is z'. For an observer at point O_1, the star is at the horizon. Clearly, our location on Earth matters for the position we observe a celestial object that is close enough.

Daily (or geocentric) parallax is the shift of a star or celestial object as seen from different points on Earth or from different times of the day.

Let us call the parallax angle $p = z' - z$. From the law of sines applied to the triangle OCS in Fig. 2.4,

$$\frac{\sin p}{a} = \frac{\sin 180 - z'}{d} = \frac{\sin z'}{d} \tag{2.6}$$

so we conclude that

$$\sin p = \frac{a}{d} \sin z' \tag{2.7}$$

The parallax is greatest where $z' = 90°$. This is called horizontal parallax, P. Its maximum is

$$\sin P = \frac{a}{d}. \tag{2.8}$$

The horizontal parallax of the Moon is

$$\sin P = \frac{a}{d} \sim \frac{1}{62} \sim 57'. \tag{2.9}$$

The parallax for an arbitrary zenital distance z' is found by substituting Eq. (2.7)

$$\sin p = \sin P \sin z'. \tag{2.10}$$

If both p and P are small angles we can approximate

$$p \approx P \sin z'. \tag{2.11}$$

Example: The Moon is occulting the star Aldebaran, with the center of the Moon coinciding with the position of the star (α = 4h 36m 00s and declination $\delta = +16°31'00''$) at culmination for a hypothetical observer at the center of the Earth. The radius of the Moon is 15′. Will an observer from APO see an occultation?

Solution: We need to find if, after correction for parallax, the disk of the Moon will be covering Aldebaran.

The geocentric altitude of the Moon is

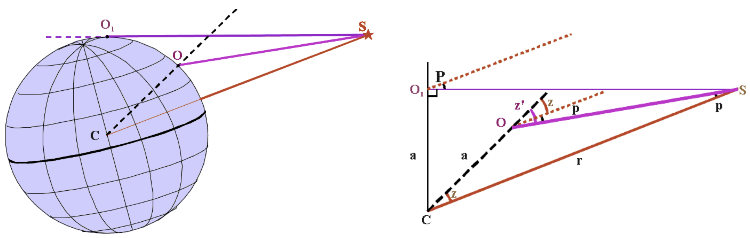

Figure 2.4: Geocentric parallax. For a hypothetical observer at the center of the Earth, a star is at zenith distance z. For an observer at O, the zenith angle is z'. For an observer at point O_1, the star is at the horizon.

$$\begin{aligned} h &= 90° - \phi + \delta \\ &= 90° - 32°47' + 16°31' \\ &= 73°44' \end{aligned}$$

So the geocentric zenith distance is $z = 90° - h = 16°16'$.

The shift by parallax is $p = P \sin z'$.

We only know z, the true zenith angle. Let us take $z \sim z' = 16°16'$ as first guess. Then

$$p = 57' \sin z' = 0.266° = 15'58'', \tag{2.12}$$

which would make the apparent zenith angle $16°16' + 16' = 16°32'$.

We can use this value of z' to refine the estimate of p. Re-calculate $p = P \sin z'$

$$p = 57' \sin(16°32') = 16'. \tag{2.13}$$

The value changed by 2 arcseconds only. Thus, we accept convergence of $p = 16'$.

We conclude that the apparent altitude of the Moon will be $16'$ lower as the altitude seen by the hypothetical observer at the center of the Earth. Since the Moon's radius is $15'$, its edge will be $1'$ away from Aldebaran. An observer at APO will not see an occultation.

2.1.3 The distance to Mars

Throughout history, determining the size of the solar system was a long-standing problem, as illustrated by the method used by Aristarchus to measure the distance to the Sun, and finding a value 20 times too short. The problem was one of precision measurement. Finally by the 1600s the introductions of telescopes allowed for extremely precise measurements of planets relative to background stars. By 1672 the precision was high enough that both Giovanni Cassini and John Flamsteed took up the challenge of measuring the distance to Mars during opposition. When a planet like Mars is at opposition, it is as close to Earth as it can possibly be, and since it is opposite the Sun it is easily seen in the dark of night. With a precise distance measurement of Mars at opposition, the scale of the solar system could be calculated.

Both Cassini and Flamsteed used parallax for their observations, but their methods were radically different. Cassini aimed to use the width of the Earth as is baseline. He sent his colleague Jean Richer to the Cayenne Island on the coast of South America, while Cassini remained in Paris. Richer arrived in Cayenne a year before the transit in order to make precise measurements of the island's latitude and longitude. Cassini and Richer then each observed the opposition of Mars, from which the parallax could be calculated.

Flamsteed's approach was to use the motion of Earth as a source of parallax. He measured the position of Mars a bit after sunset one evening, and then again a bit before sunrise the following morning. The motion of Earth during the night thus produced a parallax shift. Normally this would not be very accurate, because while Earth moves in its orbit, so does Mars, and the amount of shift in Mars' orbit would be larger than the parallax. But Flamsteed measured the parallax when Mars was at its peak of retrograde motion, so that the effects of the planets motion was quite small.

Despite their quite different methods, both Cassini and Flamsteed got a value of about 73 million km for the distance between Earth and Mars, close to the actual value of 78 million km. From Kepler's laws it was known that the distance of Mars to the Sun was 1.524 the distance from the Earth to the Sun. Therefore, the could place the Earth at 139 million km to the Sun (actual value 149 million km). It was a result that would not be improved upon until observations of a transit of Venus in 1769.

2.1.4 Annual Parallax

If the Earth moved, stars should appear to move too, due to parallax, with close stars moving more then stars further away. Indeed, for a long time, the lack of such motion was taken as evidence that the Earth did not move around the Sun.

For a star at the pole of the ecliptic, the parallax is $\tan p = \frac{a}{d}$, where a is the semimajor axis of Earth's orbit, and d the distance of the star to the Sun. Because the angle is small, we can write $\tan x \approx x$ and thus

$$d = \frac{a}{p} \tag{2.14}$$

The claimed observational accuracy of the ancients was about 5 minutes or arc (the real accuracy was between 15 and 40 minutes). Without being able to detect an annual parallax, they concluded that if the Earth moved, then the annual stellar parallax would have to be smaller than 5 minutes. That would place the sphere of the stars at a minimum distance of

$$d = \frac{1\,\mathrm{AU}}{5'} \times \frac{60'}{1^\circ} \times \frac{180^\circ}{\pi\,\mathrm{rad}} \approx 687\,AU \tag{2.15}$$

Tycho Brahe was the one who actually brought the accuracy down to this level, reaching accuracy as good as 2 minutes of arc. With that precision he could place the sphere of the stars at no closer than 1500 AU.

This distance may seems small for us, but was still too large to be acceptable, specially in light of the fact that the furthest known planet, Saturn, was at 10 AU. A giant void had to exist between the sphere of Saturn and the sphere of the stars. It was

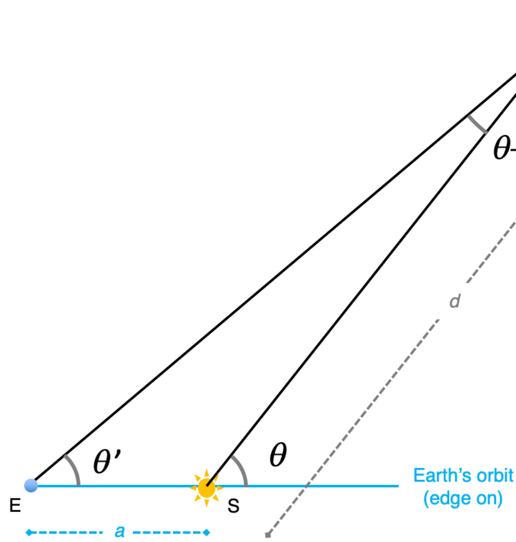

Figure 2.5: Annual parallax for a star at a random orientation with respect to the ecliptic.

possible, but aesthetiallcay displeasing, and there was no observational evidence for it. The debate was open, but until the detection of stellar parallax, unsettled.

2.1.5 Astronomical Unit and Parsec

The annual parallax invites the definition of a unit of distance. Given that trigonometrically the distance to an object is

$$d = \frac{a}{p} \tag{2.16}$$

where a is the radius of Earth's orbit and p is the annual parallax, we can set $a = 1$ and choose appropriate units to define a standard. As such, we can say that $a = 1$ defines the *astronomical unit* (symbol AU).

We can then define a distance such that the annual parallax is one arc second. This distance is called a **par**allax-**sec**ond, or *parsec* for short.

$$1 \text{ pc} = \frac{1\,AU}{1''} \tag{2.17}$$

Thus, for parallaxes measured in arcseconds, the distances in parsecs are

$$d(\text{pc}) = \frac{1}{p('')} \tag{2.18}$$

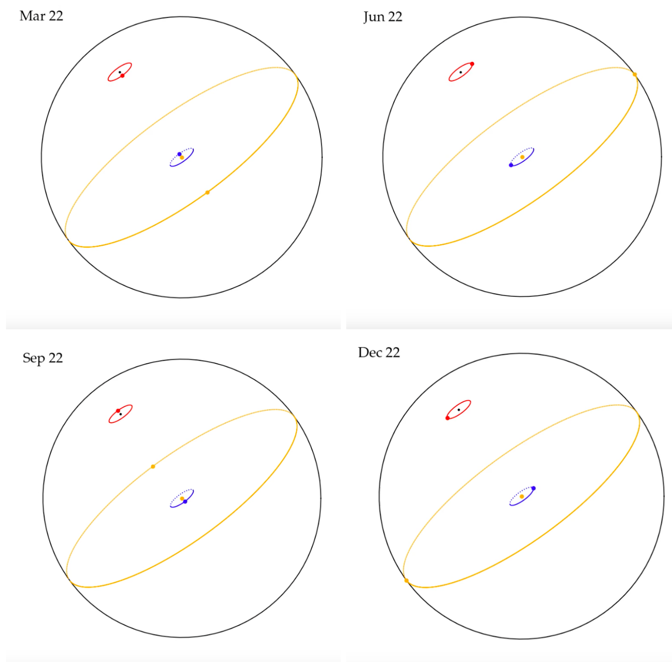

Figure 2.6: Parallax annual motion, from spring equinox (Mar 22) to winter solstice (Dec 22). The sphere is centered on the Sun, the Earth moves around in the orbit counterclockwise. The yellow circle is the ecliptic, tracing the projected position of the Sun in the sky according to our perspective. A star's true position is marked by a black dot. Because of the Earth's motion, we see the fixed star executing a reflection of the Earth's orbit, just like the Sun. This parallax shift is always in the same direction the Sun is moving.

As we will see later, the closest stellar system to the Sun is α Centauri, 1.4 pc away. The average separation between stars in the Galaxy is about a parsec.

2.1.6 The observational signature of annual parallax

Let us understand how exactly the annual parallax would appear in the sky, so that we know what to look for. For a star in a random orientation with respect to the ecliptic, the star is seen at an angle θ from the Sun, but at an angle θ' as seen from Earth (see Fig. 2.5)

Applying the law of sines to the triangle formed by the Sun, the Earth, and the star

$$\frac{\sin\theta - \theta'}{a} = \frac{\sin\theta'}{d}, \tag{2.19}$$

from which we conclude that

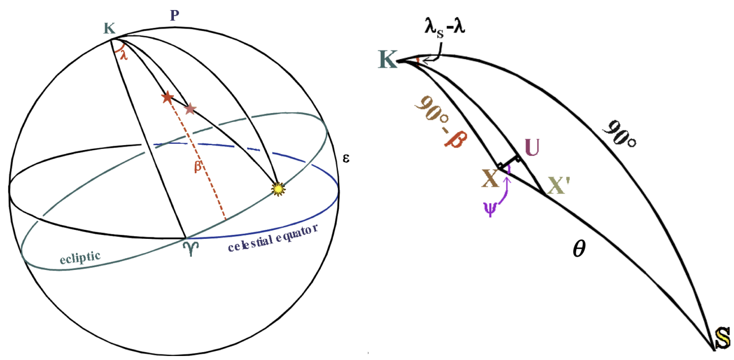

Figure 2.7: Left: A star, with ecliptic coordinates λ, β, marked by the red star, is seen to shift to a position given by the pink star. The shift is towards the Sun, because the parallax motion is a reflect of the Earth's orbital motion, the same that makes us see the Sun move across the stars. Right: We can trace a spherical triangle having vertices at the ecliptic pole K, the sun S and the actual position of the star, X. The parallax shifts the star to the position X'. The plane right triangle given by XUX' has for catheti the shifts in latitude and longitude we want to find.

$$\begin{aligned} \sin\theta - \theta' &= \sin\theta' \quad \frac{a}{d} & (2.20) \\ &= \sin\theta' \sin p. & (2.21) \end{aligned}$$

We can approximate $\theta - \theta' \ll \theta'$, so p is of the same order of $\theta - \theta'$. We can thus replace $\sin\theta - \theta'$ by $\theta - \theta'$ and $\sin p$ by p, to write

$$\theta - \theta' = p \sin\theta' \tag{2.22}$$

We are wanting to measure θ', which appears in both sides of this equation, which involves x and $\sin x$ and it therefore transcendental. But since we have established that $\theta - \theta'$ is small, we can approximate $\theta' \approx \theta$ and write

$$\theta' \approx \theta - p \sin\theta \tag{2.23}$$

So, the θ' angle measured from Earth will be the angle θ measured from the Sun, corrected by a small amount $p \sin\theta$. Since the motion is a reflexion of the Earth's motion around the Sun, the shift will always be on the direction of a line connecting the true position of the star with the position of the Sun (Fig. 2.6).

2.1.6.1 Parallax ellipse

Let us mathematize the motion. Since the shift occurs due to Earth's motion, the reference frame should be Earth's orbit, the plane of the ecliptic. The star's true coordinates are thus $X = (\lambda, \beta)$, ecliptic longitude and ecliptic latitude. Due to parallax, we see these coordinates shifted to a position $X' = (\lambda + \Delta\lambda, \beta + \Delta\beta)$ in a direction toward the Sun (Fig. 2.7). The segment XX' is the parallatic shift $p \sin\theta$.

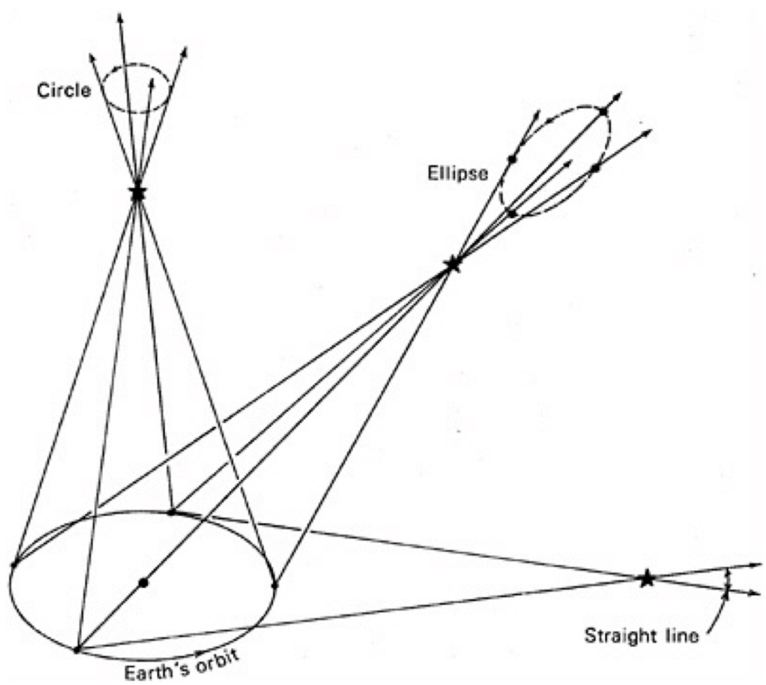

Figure 2.8: Parallax shift for stars of different eclitpic latitude. At the pole, the star executes a circle. At the equator, a line. At other latitudes it is an ellipse.

We want to find the shifts $\Delta\lambda$ and $\Delta\beta$. For that, we can define two triangles. One, a spherical triangle having vertices at the ecliptic pole K, the sun S, and the actual position of the star, X. The sides of the triangle are $KX = 90^\circ - \beta$ (from the ecliptic pole to the star), $KS = 90^\circ$ (from the ecliptic pole to the ecliptic), and $SX = \theta$ (from the Sun to the star).

The parallax shifts the star from X to the position X'. Let us define a point U at the perpendicular from X to the arc KX'. The plane right triangle given by XUX' has for catheti $UX = \Delta\lambda\cos\beta$ and $UX' = \Delta\beta$, precisely the shifts in latitude and longitude we want to find.

Define ψ as the angle $U\hat{X}X'$. Then

$$UX = XX'\cos\psi \tag{2.24}$$
$$UX' = XX'\sin\psi \tag{2.25}$$

that is

$$\Delta\lambda\cos\beta = p\sin\theta\cos\psi \tag{2.26}$$
$$\Delta\beta = p\sin\theta\sin\psi \tag{2.27}$$

We now work to eliminate θ and ψ. The spherical triangle has angles $S\hat{X}K = 90+\psi$ and $X\hat{K}S = \lambda_S - \lambda$, where λ_S is the ecliptic longitude of the Sun. From the law of sines to the spherical triangle

$$\frac{\sin(90^\circ + \psi)}{\sin 90^\circ} = \frac{\sin(\lambda_S - \lambda)}{\sin\theta} \tag{2.28}$$

that is

$$\cos\psi \sin\theta = \sin(\lambda_S - \lambda) \tag{2.29}$$

From the law of the cosines for the sides

$$\cos 90^\circ = \cos\theta\cos(90^\circ - \beta) + \sin\theta\sin(90^\circ - \beta)\cos(90 + \psi) \tag{2.30}$$

that is

$$\sin\theta\sin\psi = \frac{\cos\theta\sin\beta}{\cos\beta} \tag{2.31}$$

Use the cosine rule again to obtain $\cos\theta$

$$\cos\theta = \cos 90^\circ\cos(90^\circ - \beta) + \sin 90^\circ\sin(90^\circ - \beta)\cos(\lambda_S - \lambda) \tag{2.32}$$

that is

$$\cos\theta = \cos\beta\cos(\lambda_S - \lambda) \tag{2.33}$$

We can substitute Eq. (2.33) into Eq. (2.31) to find

$$\sin\theta\sin\psi = \cos(\lambda_S - \lambda)\sin\beta \tag{2.34}$$

Using now Eq. (2.34) and Eq. (2.29) into Eq. (2.27),

$$\begin{aligned} \Delta\lambda\cos\beta &= p\sin(\lambda_S - \lambda) & (2.35)\\ \Delta\beta &= p\cos(\lambda_S - \lambda)\sin\beta & (2.36)\\ & & (2.37) \end{aligned}$$

Since these are cosines and sines of the same angle, we square and sum the equations

$$\begin{aligned} \frac{(\Delta\lambda)^2\cos^2\beta}{p^2} &= \sin^2(\lambda_S - \lambda) & (2.38)\\ \frac{(\Delta\beta)^2}{p^2\sin^2\beta} &= \cos^2(\lambda_S - \lambda) & (2.39)\\ & & (2.40) \end{aligned}$$

i.e.

$$\frac{(\Delta\lambda)^2\cos^2\beta}{p^2} + \frac{(\Delta\beta)^2}{p^2\sin^2\beta} = 1. \tag{2.41}$$

Eq. (2.41) is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.42}$$

i.e, an *ellipse*, of parameters

$$x = \Delta\lambda \cos\beta \tag{2.43}$$
$$y = \Delta\beta \tag{2.44}$$

with semimajor and semiminor axes

$$a = p \tag{2.45}$$
$$b = p \sin\beta \tag{2.46}$$

This ellipse is the *parallatic ellipse* that the star completes over the course of the year. The semimajor axis is p, the parallax angle, and parallel to the ecliptic. On the ecliptic, $b = 0$ and therefore the star traces a line. On the ecliptic pole, $b = 90°$ and the star traces a circle (see Fig. 2.8).

Example: A star's true position is $\alpha = 6\text{h}$, $\delta = 0°$, located 25 parsecs away. On the March equinox, how far will it shift by parallax, and in what direction?

Solution: We have the equatorial coordinates of the star, and the equations in ecliptic coordinates. Let us convert the stellar position to eclliptic. Using Eq. (1.82), $\beta = -\varepsilon = -23.5°$, and using Eq. (1.83), $\lambda = \alpha = 90°$. Given the distance, the parallax in arcseconds is $p('') = 1/25\,pc = 0.04''$. At the vernal equinox, $\lambda_S = 0°$, $\lambda_S - \lambda = -90°$. We thus have

$$\Delta\lambda = \frac{p \sin(-90°)}{\cos\varepsilon} = -0.044'' \tag{2.47}$$

As for β, since $\lambda_S - \lambda = -90°$, $\Delta\beta = 0$. The star is shifted $0.044''$ westwards by parallax.

2.2 Aberration

The search for the stellar parallax was a long and harduous one in astronomy. The parallax will shift the position of a star in the same direction that the motion of the Sun happens. More northwards in June, more southwards in December, no shift in declination in the equinoxes. Yet, astronomers were baffled to find another annual motion, that behaved in the opposite way, maximizing the declination shift in the equinoxes, and zero at the solstices (Fig. 2.9). This was the *aberration of light*, an unknown phenomenon by then.

Aberration occurs because of the finite speed of light. If a source is moving with respect to us, we see it where it was when the light sent towards us. The same happens if we are moving with respect to the source. We will see it at an angle towards the direction of motion. The phenomenon is similar to walking fast in the rain – if the rain is falling vertically, you need to tilt your umbrella.

Trigonometrically, a light ray is beamed towards the moving Earth. The path is ct. In the time it took to arrive, the Earth moved by vt. Applying the law of sines to the triangle

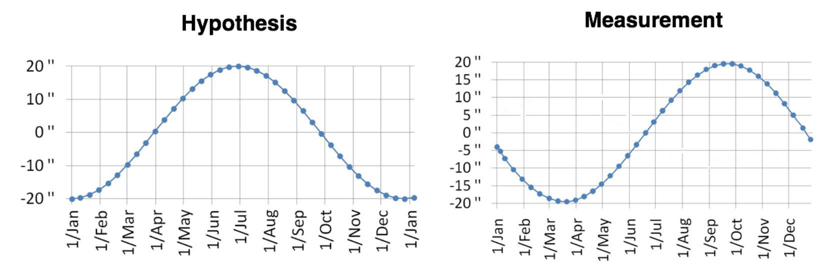

Figure 2.9: If the apparent position of a star is shifted due to parallax, the shift in declination will appear as the one shown the left plot. Just like the Sun, it will be more northermost in June, and southernmost in December. Instead, the measured motion found is the one found in the right plot. It is maximum in the equinoxes and lowest in the solstices, thus off-phase with respect to parallax. This confounded early astronomers and was dubbed aberration of light. It was finally explained by James Bradley as due to the combined effect of a finite speed of light and the motion of the Earth.

$$\frac{\sin\theta - \theta'}{vt} = \frac{\sin\theta'}{ct} \tag{2.48}$$

So

$$\theta - \theta' = \sin\theta\frac{v}{c} = k\sin\theta \tag{2.49}$$

where $k = v/c$ is the *aberration constant*. The Earth's orbital velocity being 30 km/s, then $k = 20.5''$.

Aberration also traces an ellipse as the Earth moves around the Sun. So visualize it, we consider the point toward where the velocity vector of the Earth points to (Fig. 2.10). This point is the Earth's *apex*. It is toward the apex that the aberration shift occurs (Fig. 2.11). Geometrically, the apex is always at a right angle to the Sun, trailing it (Fig. 2.10). The triangle of the problem (Fig. 2.11) is very similar that of the parallax, with only λ_S, the longitude of the Sun, being substituted by $\lambda_F = \lambda_s - 90°$, the longitude of the apex, and p by k. The equations become

$$\Delta\lambda\cos\beta = k\sin(\lambda_F - \lambda) = -k\cos(\lambda_S - \lambda) \tag{2.50}$$
$$\Delta\beta = -k\cos(\lambda_F - \lambda)\sin\beta = -k\sin(\lambda_S - \lambda)\sin\beta \tag{2.51}$$

This is an ellipse for $x = a\cos\theta$, $y = b\sin\theta$, $\theta = \lambda_S - \lambda$, semimajor axis k, parallel to the ecliptic, and semiminor axis $k\sin\beta$.

The parallax and aberration ellipses are different because aberration is much larger than parallax, and it is in a different phase.

Example: A star at α=6 h, δ=0. On the March equinox, by how much will it shift due to aberration, and in what direction?

First, convert to ecliptic coordinates, $\lambda = 90°$, and $\beta = -\epsilon = -23.5°$. At the equinox, $\lambda_S = 0$, and $\lambda_S - \lambda = -90°$. The ellipse has longitude shift

$$\Delta\lambda\cos\beta = -k\cos(\lambda_S - \lambda) = 0 \tag{2.52}$$

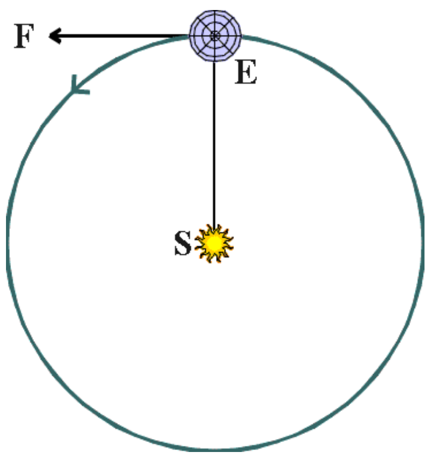

Figure 2.10: The point where the instantaneous Earth's velocity vector points to is called the Earth's *apex*. Aberration makes the apparent position of a star shift toward the apex.

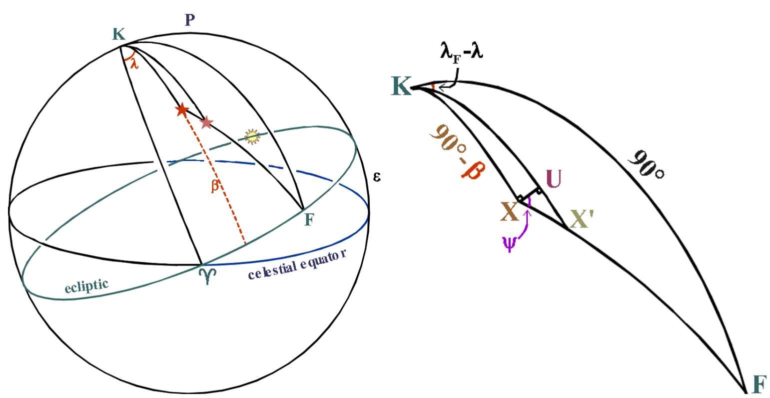

Figure 2.11: Left: A star, with ecliptic coordinates λ, β, marked by the red star, is seen to shift to a position given by the pink star. The shift is towards the Earth's apex, 90 degrees behind the Sun. The geometry is identical to the one for parallax (Fig. 2.7) except we swap the Sun by the Earth's apex. Right: We can trace a spherical triangle having vertices at the ecliptic pole K, the apex F and the actual position of the star, X. Aberration shifts the star to the position X'. The plane right triangle given by XUX' has for catheti the shifts in latitude and longitude we want to find.

So $\Delta\lambda = 0$. For latitude,

$$\Delta\beta = -k\sin(\lambda_S - \lambda)\sin\beta \tag{2.53}$$

For $k = 20.5''$ and $\beta = -23.5°$, then $\Delta\beta = +8.15''$. The star is shifted northwards, by aberration.

2.3 Proper Motion

Stars also have a motion that is not because of any motion of the Earth. This *proper motion* is called by the greek letter μ. We can decompose the motion into declination μ_δ and right ascension $\mu_\alpha \cos\delta$. Again, the correction by $\cos\delta$ is necessary because hour circles approach each other near the poles. The whole proper motion is

$$\mu = \sqrt{\mu_\alpha^2 \cos^2\delta + \mu_\delta^2} \tag{2.54}$$

2.4 Brightness

A full accounting of the brightness of stars and astronomical objects in general require an understanding of the theory of radiative transfer, that we will deal with as a separate module. In this section, we concern ourselves with basic and historical definitions.

2.4.1 Magnitudes

Astronomers use a historical scale to measure stellar brightness, which was defined in Ancient Greece (by either Hipparchus or Ptolemy, references vary), and called *magnitudes*. In Ptolemy's *Almagest*, the brightest stars are assigned first magnitude, and the faintest ones sixth magnitude, defining thus a ranking system of six categories. Being a system devised by naked eye measurements, the scale is nearly logarithmic.

Physically, when we measure how bright a star is, we are measuring the *energy flux*.

2.4.2 Flux

The flux is an intuitive measurement of energy flow. Imagine that you have a detector of area A, and you collect radiation with it, for a time interval t. If the source is constant, the amount of energy you collect will be proportional to A and to t. To be able to write a proportionality, we consider that even if the energy source is varying, during an infinitesimal time interval dt the source can be considered constant in time. Likewise, if the area is too large, different areas of the detector can also be detecting different amounts of energy, so we also consider the area to be infinitesimal. In this infinitesimal patch dA, the energy is spatially uniform. We define thus the idea of energy flowing through an area over time

$$dE \propto dA\, dt \tag{2.55}$$

and we call the proportionality factor the *flux*, giving it the symbol F

$$dE \equiv F\, dA\, dt \tag{2.56}$$

This equation[1] is in fact still missing a physical ingredient that the energy depends on. If your detector is sensitive only in, say, X-rays, but the radiation shining on it is, say, microwaves, then no energy will be measured. We conclude that we must also define the *frequency range* in which we are measuring the radiation. The *specific* flux F_ν is then the net energy flowing through per unit area per unit time in a given frequency range. We define all these intervals to be infinitesimal so that the energy is constant in space, time, and frequency ranges

$$dE_\nu = F_\nu \, dA \, dt \, d\nu \tag{2.57}$$

or

$$F_\nu \equiv \frac{dE_\nu}{dA \, dt \, d\nu} \tag{2.58}$$

The subscript ν in F_ν and dE_ν denotes that these quantities are *monochromatic*, i.e., measured in a single frequency. In practice, we cannot measure monochromatic fluxes. We always measure a finite range of frequencies (a waveband). So, when we measure flux in a given waveband, from ν_1 to ν_2, what we are measuring is

$$F = \int_{\nu_1}^{\nu_2} F_\nu d\nu. \tag{2.59}$$

The left-hand side is what is measured. The integrand in the right-hand side is mathematical abstraction. The quantity F_ν is an underlying function, an idealized monochromatic flux. So that the units of Eq. (2.59) come out correctly, F_ν must be defined per hertz. The unit[2] of monochromatic flux F_ν is thus

$$[F_\nu] = \text{energy (time)}^{-1} \text{ (area)}^{-1} \text{ (frequency)}^{-1} \tag{2.60}$$

$$= \text{erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1} \tag{2.61}$$

whereas the unit of flux F is

$$[F] = \text{energy (time)}^{-1} \text{ (area)}^{-1} \tag{2.62}$$

$$= \text{erg s}^{-1} \text{ cm}^{-2}. \tag{2.63}$$

We can also define the *bolometric flux*, which is simply the total flux, integrated in all frequencies.

$$F_{\rm bol} = \int_0^\infty F_\nu \, d\nu \tag{2.64}$$

2.4.2.1 The flux follows an inverse square law

Considering an isolated star, if we put spherical surfaces s and $\mathcal{S}$ of radius r and R around it (Fig. 2.12, left), by conservation of energy the total energy passing through s and $\mathcal{S}$ must be the same. Thus,

[1] A note on notation: as there are two infinitesimals in the right-hand side, the left-hand side should not be dE but d^2E. In practice, radiative transfer theory tends to not use this more rigorous notation, and exponents in infinitesimals are dropped.

[2] In astronomy we often use cgs units instead of SI, which you may be more used to. In cgs, the unit of length is *centimeter* (1 cm = 10^{-2} m), the unit of mass is *gram* (1 g = 10^{-3} kg), and the unit of time is *second*, as in SI. The unit of energy, the *erg*, is built from these, being erg ≡ g cm^2/s^2. The equivalence between that and SI is 1 erg = 10^{-7} J. A list of units and constants in cgs is shown in the end of this chapter

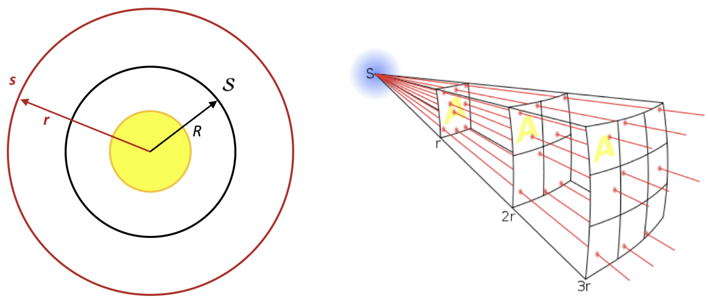

Figure 2.12: *Left*: The energy leaving the stellar surface is the same energy radiated through the surfaces $\mathcal{S}$ and s of radii R and r, respectively. *Right*: At greater radii, the same amount of energy disperses through a larger area. The flux (energy per area) thus decreases at the same rate that the area increases, with the square of the distance.

$$F(r)\,4\pi r^2 = F(R)\,4\pi R^2 \tag{2.65}$$

or

$$F(r) = F(R)\left(\frac{R}{r}\right)^2 \tag{2.66}$$

If we consider R instead as the radius of the star and r an arbitrary location away from the stellar surface, Eq. (2.66) says that the flux falls with the square of the distance. This is merely a statement of conservation of energy (Fig. 2.12, right).

2.4.3 Flux and magnitude

In linear scale, the difference in flux between a star of 1st and 6th magnitude is approximately 100. In the 19th century, the astronomer Norman Pogson suggested to make this a standard, i.e., to fix a difference of 5 magnitudes as a factor 100 in flux

$$m_1 - m_2 = 5 \quad \Longleftrightarrow \quad \frac{F_2}{F_1} = 100 \tag{2.67}$$

Therefore, the magnitude scales with the flux acc ording to

$$m = -2.5\log F + C \tag{2.68}$$

where C is a constant. The constant is arbitrarily chosen so that the magnitude of the star Vega is zero.

2.4.4 Apparent and Absolute magnitudes

Because the flux depends on distance d following an inverse square law

$$F \propto \frac{1}{d^2}, \tag{2.69}$$

the magnitudes we measure are also a function of distance. For this reason, we call them *apparent magnitudes*.

We could define *absolute* magnitudes, independent of distances, that would reflect a star's true brightness. For that we need a standard distance D, that was arbitrarily defined as $D = 10\,\mathrm{pc}$. The absolute magnitude M is thus

$$M = -2.5 \log \left[F(10pc)\right] + C \tag{2.70}$$

Notice that the difference between apparent and absolute magnitude

$$m - M = -2.5 \log \left[\frac{F(d)}{F(D)}\right] \tag{2.71}$$

is a quantity that depends only on the distance. For this reason, $m - M$ is also called the *distance modulus*. Because the flux follows an inverse square law, the distance modulus can be written as

$$m - M = 5 \log \left(\frac{d}{10\mathrm{pc}}\right) \tag{2.72}$$

The distance, in parsecs, is then given by

$$d(\mathrm{pc}) = 10^{0.2(m-M)+1} \tag{2.73}$$

It is also useful to express this as

$$M = m + 5 + 5 \log \pi'' \tag{2.74}$$

where $\pi = 1/d(\mathrm{pc})$, measured in arcseconds, is the parallax angle (do not confuse it with the circle constant!). The equation above depends only on measurable quantities and is useful to have in handy when observing.

2.4.5 Luminosities

We define the bolometric flux as the flux integrated in all wavelengths (Eq. 2.64).

$$F_{\mathrm{bol}} = F = \int_0^\infty F_\nu \, d\nu \quad \text{and} \quad F = \frac{dE}{dAdt}$$

The quantity dE/dt in physics is called power, yet in astronomy we prefer to call it *luminosity*. So, Flux × Area = Luminosity. For a source of constant luminosity, as most sources in astrophysics are (in the timescales we measure them), the product Flux × Area is constant. Therefore, the flux falls with area following an inverse square law, recovering Eq. (5). For a spherical star, the area is $4\pi R^2$ and we can write

$$L_\star = \text{Area} \times \text{Flux} = 4\pi R^2 \, F_\star . \tag{2.75}$$

where $F_\star$ is the flux at the stellar surface.

2.4.6 Tying magnitudes and luminosities: bolometric correction

Notice that in Eq. (2.74) we wrote flux instead of luminosity, even though the idea of an absolute magnitude is to resolve the distance degeneracy so that we measure the stars' true brightness relative to one another.

The reason is that luminosity corresponds to the *bolometric* flux of the star, whereas magnitudes are usually measured within a specific range of wavelengths (a waveband). Hipparchus and Ptolemy, working with the naked eye, did not have means to measure infrared or ultraviolet magnitudes. They measured it in the visible part of the electromagnetic spectrum only. Thus, these are also called *visible* magnitudes. We can call the apparent visible magnitude V, and the absolute visible magnitude M_V.

A *bolometric correction* is needed to convert these magnitudes to a bolometric magnitude that covers the whole spectrum and can be tied to luminosity.

$$M_{\mathrm{bol}} = M_V + BC \tag{2.76}$$

For the Sun, the absolute visible magnitude is $M_V = 4.83$. The Sun's bolometric correction is BC=-0.09. The Sun has but a marginal bolometric correction because most of its energy is radiated in the visible. Stars much cooler or much hotter than the Sun can have significant bolometric corrections as most of their energy is radiated in other parts of the electromagnetic spectrum.

Having bolometric magnitudes, we can write

$$M_{\mathrm{bol}} = -2.5 \log\left(\frac{L_\star}{4\pi D^2}\right) + C \tag{2.77}$$

And thus, in solar units, the stellar luminosity is

$$\frac{L}{L_\odot} = 10^{-0.4(M_{\mathrm{bol}} - M_{\odot,\mathrm{bol}})} \tag{2.78}$$

Several wavebands are in use in astronomy. One of the most popular waveband system is the UBV system of Jonhson.

Problems

1. A minor planet (asteroid) passes very near the Earth, at a distance of 200,000 km.
   (a) What will be its horizontal parallax? (Take the Earth to be a sphere of radius 6378 km.)
   (b) At latitude 56°, the minor planet is observed to cross the meridian at an apparent altitude of 35°. What does its declination appear to be?
   (c) What is its true declination, after correcting for geocentric parallax?
2. The Moon is at declination −14°. What will be its hour angle at moonrise (when the top edge of the Moon first appears over the horizon), at a latitude of 34°?
3. Given the parallax shifts $\Delta\lambda$ and $\Delta\beta$, find the parallax shifts in celestial equatorial coordinates $\Delta\alpha$ and $\Delta\delta$
4. A star's true position is α=8 h and $\delta = -30°$, 10 pc away. On the June solstice, how far will the star shift by parallax, and on what direction?

5. The star Spica (α Vir) is at 77 pc away from the Sun.
   (a) What is its parallax seen from Earth?
   (b) What is its parallax using as base the orbit of Neptune, at 30 AU?

6. Show that the tangential velocity of a star, in km/s, is given by

$$v_t = \frac{4.75\mu}{p} \tag{2.79}$$

   where μ is the proper motion in arcsec per year, and p is the parallax in arcseconds.

7. In 1672, Cassini determined the distance to Mars by measuring its position in the sky from Paris (48°51′N, 2°20′E), while his assistant, Jean Richer, measured the position of Mars at the same time from Cayenne (4°55′N, 52°18′W), in the French Guiana. By comparing their measurements, they could determine the parallax shift of Mars relative to distant stars.
   (a) Show that the great circle distance from Paris to Cayenne is approximately 7 000 km.
   (b) What is the *linear* distance between Paris and Cayenne? The linear distance is the length of the chord that crosses the interior of the Earth connecting the two cities.
   (c) The parallax found was 9″. What is the distance to Mars?

8. The equatorial coordinates of Procyon (α Canis Minoris) are α = 7h 39m 18s, δ = 05°13′30″. The components of its proper motion are $\mu_\alpha = -0.0476$ s/yr, $\mu_\delta = -1.037''$/yr. Its parallax is 0.284″, and radial velocity -3.2 km/s.
   (a) What are the tangential and total velocities of Procyon?
   (b) The total velocity is the 3D velocity that Procyon is moving through the Galaxy, with respect to the Sun. Assuming that this motion is uniform, when will Procyon be physically closest to the Sun?
   (c) What will its proper motion and parallax be then?

9. You are an astronaut sent on a space mission. Waking up from your sleeping pod after an unknown travelled distance and an unknown elapsed time, you find yourself on an alien world. Not knowing where you are, you do what an astronomer would do, and start observing the skies. You choose a star and measure its position periodically, with accuracy of 0.1″. You detect a full period of aberration, shown in the upper plot of Fig. 2.13. Yet, the data seems to be wobbling back and forth, so you think that there must be another signal in the data. You isolate the first signal (middle plot) and subtracting it from the data, you measure what seems to be another oscillation, of much smaller amplitude, and a different period (lower plot).

   Measure from Fig. 2.13 the period (in days) and amplitude (in arcsecs) of the primary and the secondary signals. If the secondary signal is also from aberration, what can you conclude about the orbital configuration of the world you find yourself at?

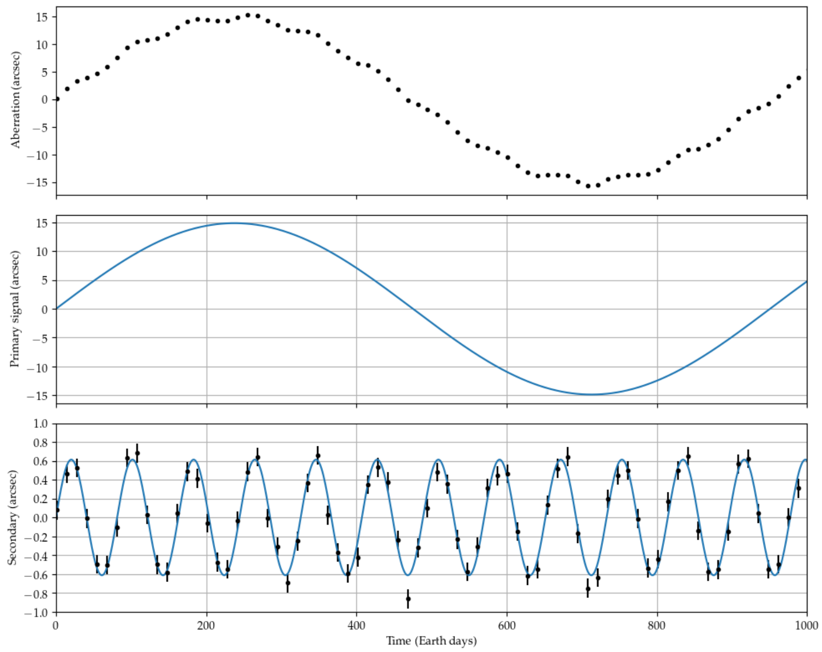

Figure 2.13: Careful observations of a star from the surface of a hypothetical world reveal an aberration shift, shown in the upper plot. After subtracting the primary signal (middle plot), a secondary periodic signal of smaller amplitude is found (lower plot).

10. Find the orbital velocities and orbital radii related to the two motions you found in the previous exercise.

11. Derive Eq. (2.68) from Eq. (2.67).

12. Given that magnitude and flux are related by $m = -2.5 \log_{10} F + C$, what flux ratio is equivalent to the difference of one magnitude? Round the answer to three decimal points.

13. Two stars, at a distance of 1.3 parsecs from the Sun, form a binary system with a mean separation of 20 AU, similar to that of Uranus from the Sun.

    (a) What is the projected separation of the stars measured from Earth, in arcseconds?

    (b) The theoretical resolving power of a telescope, in radians, is $1.2\lambda/D$, where λ is the wavelength of light, and D (expressed in the same units as λ) is the diameter of the objective lens or mirror. What is the minimum diameter for the objective of a telescope that can reveal the system as a visual binary, in centimeters? Assume 5500Å for the center of the visible spectrum. Knowing that the diameter of the eye's pupil is about 5 mm, can you resolve it with the naked eye?

(c) Assuming that one star has magnitude 0 and the other one has magnitude 1, what is the magnitude you see with the naked eye? Warning: do not overlook the constant. Round the answer to two decimal points.

(d) What are the stars' absolute magnitudes?

(e) Knowing that the Sun's visual magnitude is -26.7, and assuming that the stars have equal bolometric corrections, what is the luminosity of each star, compared to the Sun's?

14. Show that, in general, the combined magnitude of a system of N objects is

$$M_{\text{total}} = -2.5 \log_{10} \sum_{i}^{N} 10^{-0.4 m_i} \tag{2.80}$$

where m_i is the magnitude of each individual i-th object.

15. The Earth and the Moon are separated by an average distance of $\approx$ 240 thousand miles, or $\approx 3.8 \times 10^{10}$ cm.

(a) What is the maximum angular separation between the Earth and the Moon as seen in the Martian sky, measured in arcminutes? Mars is at $\approx$1.4 AU from the Sun.

(b) If you stood on Mars, could you resolve the Earth-Moon system with with the naked eye?

(c) The Earth is seen from Mars with a magnitude of -2.5. The Moon is seen from Mars with magnitude 0.9. What is the magnitude an observer on Mars will measure if the Moon and the Earth are not resolvable? Round the answer to two decimal points.

16. The Andromeda galaxy is visible in the night sky as a diffuse cloud of 4th magnitude ($V = 3.44$, where V stands for magnitude in the visible waveband.)

(a) Several ways to estimate the distance to Andromeda agree on the value of $\approx$ 780 kpc. Show that the absolute magnitude of the Andromeda galaxy in the same band is $M_V \approx -21$.

(b) Assume that the radiation coming from Andromeda is all due to solar-like stars ($M_{V,\odot} = 4.83$). How many stars are there in the Andromeda galaxy according to this approximation?

(c) The actual number of stars in the Andromeda galaxy is in fact about a trillion. Based on that and your answer to the previous question, what can you conclude about the luminosity of the typical star in that galaxy?

(d) The most common stars in the solar neighborhood are M dwarfs. Let us assume that the same holds true in other galaxies. M dwarfs range in mass from 0.08 to 0.5 $M_\odot$, where $M_\odot$ is the mass of the Sun. The mass-luminosity relationship for main sequence stars can be approximated as $L/L_\odot = (M/M_\odot)^{3.5}$. With these new information, refine your estimate of the number of stars in the Andromeda galaxy. Assume that a characteristic M dwarf has a third of the mass of the Sun