Chapter 4

Celestial Mechanics

4.1 Gravitation

Kepler 3rd law implies something interesting about the velocity of the planets. Consider a planet in a circular orbit of radius r. The circumference is $C = 2\pi r$. The planet traverses the orbit with velocity v in a period T. Equating $C = vT$,

$$T = \frac{2\pi r}{v} \tag{4.1}$$

For constant v, we would have $T \propto r$. The only way this relation can agree with Kepler's 3rd law, $T \propto r^{3/2}$, is if the velocity is proportional to $1/\sqrt{r}$.

Further from the Sun, a planet's period is longer not only because it has a longer track to run, but also because it is going *slower*.

Kepler realized that his law suggests that whatever keeps the planets in motion is something that emanates from the Sun and wanes with distance, like light.

Let us imagine then a "gravity luminosity", L_g, emanating from the Sun at a constant rate (Fig. 4.1). As it flows outwards, it spreads itself over progressively larger areas. At any particular distance r from the source, the "gravity flux" F_g obeys

$$L_g = F_g \times 4\pi r^2 \quad \longleftrightarrow \quad F_g = \frac{L_g}{4\pi r^2} \tag{4.2}$$

The strength of gravity should be related to the flux of these mysterious entities emanating from the Sun. Whatever it is, its strength must fall with the square of the distance, i.e.

$$\boxed{F_g \propto \frac{1}{r^2}} \tag{4.3}$$

This is the origin of the "inverse square law". A geometrical dilution effect, rooted on the fact that the area of the sphere is proportional to the square of the radius. Ultimately, the inverse square law is a consequence of the Universe being threedimensional. Such insight predates Isaac Newton.

Kepler's 3rd law has other consequences. If we consider $T = 2\pi/\Omega$, then the law implies that

$$\frac{4\pi^2}{\Omega^2} \propto r^3. \tag{4.4}$$

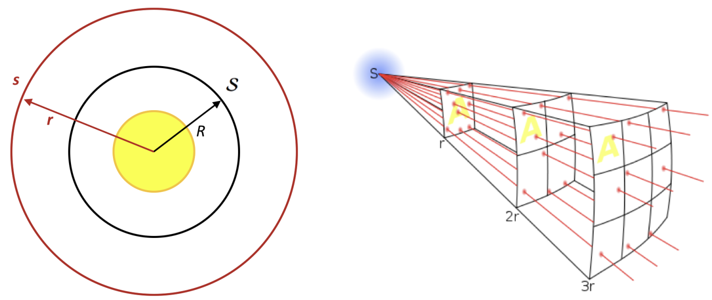

Figure 4.1: *Left*: The amount of light leaving the stellar surface is the same amount radiated through the surfaces $\mathcal{S}$ and s of radii R and r, respectively. *Right*: At greater radii, the same amount of light disperses through a larger area. The flux thus decreases at the same rate that the area increases, with the square of the distance. The same geometric dilution effect applies to gravity.

Multiplying both sides by Ω^2,

$$\Omega^2 r^3 \propto 4\pi^2 \equiv \text{const} \tag{4.5}$$

So, the left hand side is constant. Dividing by r^2,

$$\Omega^2 r \propto \frac{1}{r^2} \tag{4.6}$$

we find that what falls with r^2 is $\Omega^2 r$, which we know is the modulus of the centrifugal acceleration. Newton also knew this was the expression of the centrifugal acceleration, as derived by Christiaan Huygens in his analysis of the pendulum.

Could it be that gravity and the centrifugal force were identical? That's a question that intrigued Isaac Newton. Considering the motion of the Moon, the centrifugal acceleration it is subject to is

$$\Omega^2 r = \frac{v^2}{r} \approx \frac{(1\ \text{km/s})^2}{400\,000\ \text{km}} = 2.5 \times 10^{-3}\ \text{m/s}^2 \tag{4.7}$$

As for the gravity on the Moon, it should be the gravity on the surface of the Earth, $g_\oplus = 9.8\,\text{m/s}^2$ scaled by the square of the distance to the Moon

$$g = g_\oplus \left(\frac{R_\oplus}{r}\right)^2 \approx 9.8\,\text{m/s}^2 \left(\frac{1}{60}\right)^2 \approx 2.5 \times 10^{-3}\ \text{m/s}^2 \tag{4.8}$$

Coincidences in science are suspicious. The conclusion is momentuous: the force $g_\oplus$ that makes stuff fall on Earth is the same force that keeps the Moon in its orbit.

The full expression of this force of gravity is found when Newton's laws of motion are considered. First, the force is vectorial. It is radial, attracting the body to the Sun. Its vector expression is

$$\boldsymbol{F}_g \propto -\frac{1}{r^2}\hat{\boldsymbol{r}} \tag{4.9}$$

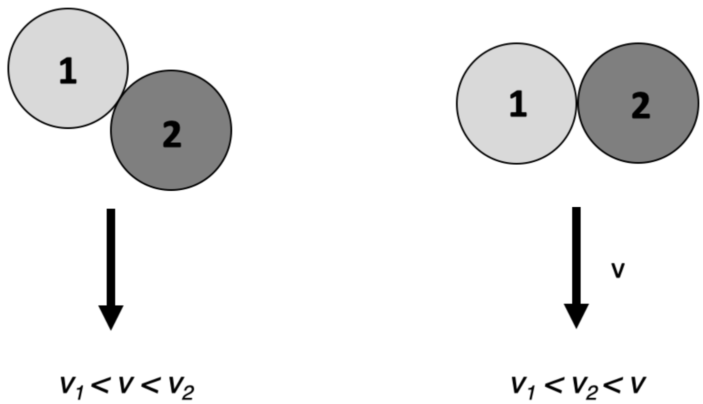

Figure 4.2: Galileo thought experiment suggesting that all bodies must fall with the same acceleration.

According to the 2nd law, a body of mass m_1 is accelerated by a force $\boldsymbol{F}_1$ according to

$$\boldsymbol{F}_1 = m_1 \boldsymbol{a}_1 \tag{4.10}$$

So, the acceleration $\boldsymbol{a}_1$ is

$$\boldsymbol{a}_1 = \frac{\boldsymbol{F}_1}{m_1} \tag{4.11}$$

Yet, if $\boldsymbol{a}$ is caused by gravity, Galileo's reasoning and experiments had already shown that bodies of different mass fall with the same acceleration. Galileo arrived at this conclusion by an ingenious thought experiment (Fig. 4.2). Imagine two spheres, 1 and 2, of different masses, with $m_1 < m_2$. If the heavier one fell faster, than if you attach them together, sphere 1 will slow down sphere 2. The resulting velocity will be between the velocities v_2 and v_1 that the spheres would attain if released individually. That's the situation sketched in the left hand side of Fig. 4.2. Yet, you can also imagine that the two spheres, attached, result in a single object of mass $m = m_1 + m_2$. In this case, the velocity that it should attain is bigger than either v_2 or v_1. The only way to reconcile the paradox is to have $v = v_2 = v_1$, i.e., bodies of different mass attain the same velocity and are thus suject to the same acceleration.

So, Eq. (4.11) must be independent of mass. The only way to have $\boldsymbol{F}_g/m$ not depending on mass is if the force of gravity is proportional to the mass.

$$\boldsymbol{F}_1 \propto -\frac{m_1}{r^2}\hat{\boldsymbol{r}} \tag{4.12}$$

According to Newton's 3rd law, if a body feels this gravity, then the source of gravity feels an equal and opposite law $\boldsymbol{F}_2 = -\boldsymbol{F}_1$, i.e.,

$$\boldsymbol{F}_2 \propto \frac{m_1}{r^2}\hat{\boldsymbol{r}} \tag{4.13}$$

Yet, $\boldsymbol{F}_2$ must also obey Newton's 2nd law,

$$\boldsymbol{F}_2 = m_2\boldsymbol{a}_2 \tag{4.14}$$

The acceleration $\boldsymbol{a}_2$, because it is gravitational, must also be independent of the mass m_2. So, F_2 must depend linearly on m_2

$$\boldsymbol{F}_2 \propto \frac{m_2}{r^2}\hat{\boldsymbol{r}} \tag{4.15}$$

The only way to make Eq. (4.13) and Eq. (4.15) compatible is if the gravitational force is proportional to both m_1 and m_2

$$\boxed{\boldsymbol{F}_g = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}} \tag{4.16}$$

The proportionality constant G is the universal gravitational constant. Equation 4.16 is known as the universal law of gravitation. The value of G was measured later to be $G = 6.67408 \times 10^{-8}\ \mathrm{cm}^2\ \mathrm{g}^{-1}\ \mathrm{s}^{-3}$.

4.2 From Newton to Kepler

Let us consider Newton's 2nd law

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} \tag{4.17}$$

and consider the motion of a planet. The acceleration $\ddot{\boldsymbol{r}}$ is gravity. Let us use a coordinate system centered on the Sun so that gravity must always point in the radial direction. The acceleration is

$$\ddot{\boldsymbol{r}} = -\frac{GM_\odot}{r^2}\hat{\boldsymbol{r}} \tag{4.18}$$

where $M_\odot$ is the mass of the Sun. Let us consider the radial position vector

$$\boldsymbol{r} = r\hat{\boldsymbol{r}} \tag{4.19}$$

Its derivative is

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}} \tag{4.20}$$

The radial and azimuthal unit vectors are a rotation of the Cartesian unit vectors

$$\begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = R_\phi \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix} \tag{4.21}$$

with the rotation matrix R_ϕ

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{4.22}$$

So, the unit vectors are

$$\hat{\boldsymbol{r}} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}} \tag{4.23}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}} \tag{4.24}$$

We want to know $\dot{\hat{\boldsymbol{r}}}$. The unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are fixed in space, but the angle $\phi(t)$ is a function of time. The derivative of $\hat{\boldsymbol{r}}$ is

$$\dot{\hat{\boldsymbol{r}}} = -\sin\phi\dot{\phi}\hat{\boldsymbol{x}} + \cos\phi\dot{\phi}\hat{\boldsymbol{y}} \tag{4.25}$$
$$= \dot{\phi}\hat{\boldsymbol{\phi}} \tag{4.26}$$

Substituting this on Eq. (4.20) we find the expression for the velocity

$$\boxed{\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}} \tag{4.27}$$

For the acceleration, we need to take the derivative of Eq. (4.27)

$$\ddot{\boldsymbol{r}} = \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\hat{\boldsymbol{r}}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}\dot{\hat{\boldsymbol{\phi}}} \tag{4.28}$$

The derivative of $\hat{\boldsymbol{\phi}}$ is

$$\dot{\hat{\boldsymbol{\phi}}} = -\cos\phi\dot{\phi}\hat{\boldsymbol{x}} - \sin\phi\dot{\phi}\hat{\boldsymbol{y}} \tag{4.29}$$
$$= -\dot{\phi}\hat{\boldsymbol{r}} \tag{4.30}$$

So Eq. (4.28), the acceleration, becomes

$$\boxed{\ddot{\boldsymbol{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}}} \tag{4.31}$$

These are two expressions. One for the radial acceleration, one for the azimuthal. Comparing Eq. (4.33) with Eq. (4.18). We have

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM_\odot}{r^2} \tag{4.32}$$
$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \tag{4.33}$$

Let us examine the azimuthal equation first. This equation is of the form $A\dot{B} + \dot{A}B$, which means that it can be written as the time derivative of AB. Indeed,

$$\frac{d}{dt}\left(r^2\dot{\phi}\right) = r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} \tag{4.34}$$

So the 2nd equation is

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 \tag{4.35}$$

The quantity $L = r^2\dot{\phi}$ is the modulus of the angular momentum, $\boldsymbol{L} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$. This equation implies that the magnitude of the angular momentum vector is conserved

$$\dot{L} = 0 \tag{4.36}$$

Notice also that for an infinitesimal time interval dt, a planet moves in its orbit by an infinitesimal length $rd\phi$. The area swept by the radius vector is thus

$$dA = \frac{r^2}{2}d\phi \tag{4.37}$$

Divide it by dt

$$\dot{A} = \frac{1}{2}r^2\dot{\phi} \tag{4.38}$$

We can now take a second derivative

$$\frac{d\dot{A}}{dt} = \frac{d}{dt}\left(\frac{1}{2}r^2\dot{\phi}\right) \tag{4.39}$$

And because of Eq. (4.35), this equals zero. That means that $\dot{A} \equiv$ const, i.e., a planet sweeps equal areas in equal times, proving, through dynamics, Kepler's 2nd law.

4.2.1 Ellipses

Consider the individual motion of the bodies 1 and 2. The equations of motion are

$$\begin{aligned} \boldsymbol{F}_1 &= \frac{Gm_1m_2}{r^3}\boldsymbol{r} = m_1\ddot{\boldsymbol{r}}_1 && (4.40) \\ \boldsymbol{F}_2 &= -\frac{Gm_1m_2}{r^3}\boldsymbol{r} = m_2\ddot{\boldsymbol{r}}_2 && (4.41) \\ & && (4.42) \end{aligned}$$

We subtract one from the other and write

$$\begin{aligned} \ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{r}}_2 - \ddot{\boldsymbol{r}}_1 && (4.43) \\ &= \frac{\boldsymbol{F}_2}{m_2} - \frac{\boldsymbol{F}_1}{m_1} && (4.44) \\ &= -\frac{G(m_1+m_2)}{r^2}\hat{\boldsymbol{r}} && (4.45) \end{aligned}$$

we can define the quantity $\mu = G(m_1 + m_2)$ and write the radial equation of motion as

$$\ddot{r} - r\dot{\phi}^2 = -\frac{\mu}{r^2} \tag{4.46}$$

Finding the orbit, that is, the equation for the position radius $r(\phi, t)$, means integrating Eq. (4.46).

To do so, we define the variable $u = 1/r$. The time derivative of r is thus

$$\begin{aligned} \dot{r} &= \frac{d}{dt}\left(\frac{1}{u}\right) && (4.47) \\ &= -\frac{1}{u^2}\frac{du}{d\phi}\dot{\phi} && (4.48) \\ &= -r^2\dot{\phi}\frac{du}{d\phi} && (4.49) \end{aligned}$$

The factor in front of the derivative is the angular momentum, $L = r^2\dot{\phi}$, so

$$\dot{r} \quad = \quad -L\frac{du}{d\phi} \tag{4.50}$$

But L is constant, so

$$\ddot{r} \quad = \quad -L\frac{d^2u}{d\phi^2}\dot{\phi} \tag{4.51}$$

We now substitute $\dot{\phi} = L/r^2$ to write

$$\ddot{r} \quad = \quad -\frac{L^2}{r^2}\frac{d^2u}{d\phi^2}. \tag{4.52}$$

We can now use this result to replace $\ddot{r}$ in the radial equation of motion Eq. (4.46)

$$-\frac{L^2}{r^2}\frac{d^2u}{d\phi^2} - r\dot{\phi}^2 = -\frac{\mu}{r^2} \tag{4.53}$$

Multiplying all terms by -1, replacing again $\dot{\phi} = L/r^2$, and multiplying all terms by r^2/L^2, we arrive at the equation of the orbit

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{\mu}{L^2}} \tag{4.54}$$

This is an ordinary differential equation of the harmonic type

$$\frac{d^2y}{dx^2} + y = C \tag{4.55}$$

whose general solution is

$$y = C\left[1 + A\cos\left(x - x_0\right)\right] \tag{4.56}$$

where A and x_0 are integration constants. Inverting the solution from $y = u$ to $u = 1/r$,

$$r = \frac{L^2/\mu}{1 + A\cos(\phi - \phi_0)} \tag{4.57}$$

Comparing it with the Kepler ellipse solution

$$r = \frac{a(1 - e^2)}{1 + e\cos f} \tag{4.58}$$

we recognize

$$L^2/\mu \quad = \quad a(1 - e^2) \tag{4.59}$$

$$A \quad = \quad e \tag{4.60}$$

$$\phi - \phi_0 \quad = \quad f \tag{4.61}$$

ϕ is the longitude, and ϕ_0 is the longitude of periastron.

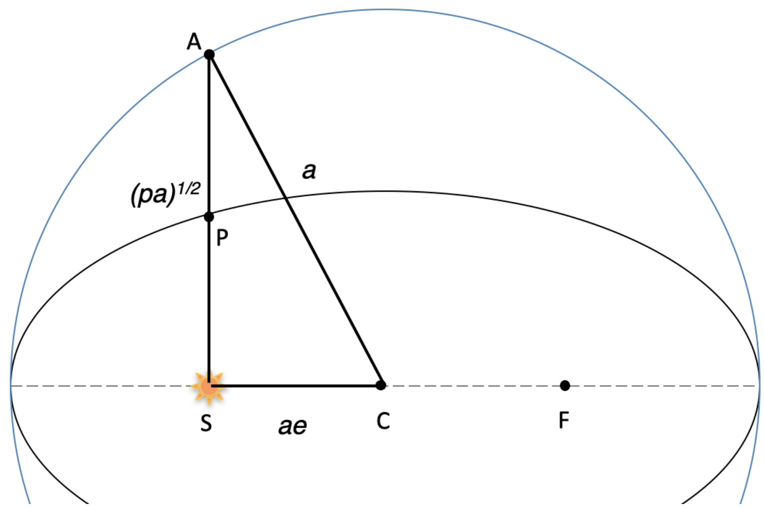

Figure 4.3: The semilatus rectum of the ellipse $p = b^2/a = a(1 - e^2)$ is the side SP.

4.2.1.1 Semilatus rectum

The quantity $p = a(1 - e^2) = L^2/\mu$ depends on angular momentum alone and is therefore a constant.

Multiplying it by a and expanding the LHS

$$pa = a^2 - (ae)^2 \tag{4.62}$$

This has the functional expression of the Pythagoras triangle, where the hypothenuse is a and one the catheti ae. Drawing an ellipse (Fig. 4.3) with foci S and F, with center C, the side $\overline{\mathrm{SA}}$ where A is the point in the circumscribed circle perpendicular to the line of apsides from S is $\sqrt{pa}$.

Because of the property of the ellipse that

$$\frac{\overline{\mathrm{SP}}}{\overline{\mathrm{SA}}} = \frac{b}{a} \tag{4.63}$$

we have that

$$\overline{\mathrm{SP}} = \sqrt{pa}\frac{b}{a} \tag{4.64}$$

We notice also that the semiminor axis of the ellipse is $b = a\sqrt{1 - e^2}$, so $b^2 = pa$ and thus

$$\overline{\mathrm{SP}} = p \tag{4.65}$$

The quantity p is the size of the line segment perpendicular to the focus S to the ellipse P. This is called the *semilatus rectum* of the ellipse (from Latin, semilatus, half-side, and rectum, perpendicular).

4.2.1.2 3rd law

Considering the equation for angular momentum conservation

$$\frac{dA}{dt} = \frac{L}{2} \equiv \text{const} \tag{4.66}$$

We can integrate it to find

$$\Delta A = \frac{L}{2}\Delta t \tag{4.67}$$

For one full period T, the area is $A = \pi ab\ T$

$$\pi ab = \frac{LT}{2} \tag{4.68}$$

Isolating T and squaring it

$$T^2 = \frac{4\pi^2 a^2 b^2}{L^2} \tag{4.69}$$

We now use $L^2 = p\mu$

$$T^2 = \frac{4\pi^2 a^2 b^2}{p\mu} \tag{4.70}$$

And, further, $b^2 = a^2(1-e^2)$, and $p = a(1-e^2)$, eliminating the eccentricity, to find

$$\boxed{T^2 = \frac{4\pi^2}{\mu}a^3}, \tag{4.71}$$

which is Kepler's 3rd law.

4.3 Constants of motion

A planet in an orbit is a motion of six variables: three components of position and three components of velocity. We need therefore six constants to be able to integrate the motion. So far we identified the magnitude of the angular momentum vector as constant, which we used to show Kepler's three laws. Let us see what other quantities are found to be constant.

4.3.1 Angular momentum

The angular momentum is in fact a vector

$$\boldsymbol{L} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{4.72}$$

which is perpendicular to the position vector and to the linear momentum. Taking the time derivative

$$\dot{\boldsymbol{L}} = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} \tag{4.73}$$

The first term zeroes because it is a cross product of a vector by itself. The acceleration $\ddot{\boldsymbol{r}}$ is given by the equation of motion, yielding

$$\dot{\boldsymbol{L}} = -\frac{\mu}{r^3}\boldsymbol{r} \times \boldsymbol{r} = 0. \tag{4.74}$$

i.e., not only the magnitude, but the three components of the angular momentum vector are conserved. This yields three constants of motion. Another conclusion that we can draw from this is that, because $\boldsymbol{L}$ is fixed and perpendicular to the position vector, the motion is restricted to a planet perpendicular to $\boldsymbol{L}$.

4.3.2 Energy

Consider the dot product of velocity and acceleration, $\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}$. Substituting Eq. (4.27) and Eq. (4.33)

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = \left(\dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}\right) \cdot \left(-\frac{\mu}{r^2}\hat{\boldsymbol{r}}\right) = -\frac{\mu\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{\mu}{r}\right) \tag{4.75}$$

Yet, we can also write

$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = \frac{1}{2}\frac{d}{dt}\left(\dot{r} \cdot \dot{r}\right) = \frac{1}{2}\frac{dv^2}{dt} \tag{4.76}$$

Equating Eq. (4.75) and Eq. (4.76), we find

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right) = 0 \tag{4.77}$$

So,

$$\boxed{\frac{1}{2}v^2 - \frac{\mu}{r} \equiv \text{const}} \tag{4.78}$$

Equation 4.78 is the *energy*. In this form, it has historically been called the *vis viva integral*.

4.3.3 Eccentricity (Runge-Lenz) vector

Consider now the vector product of the angular momentum and the acceleration, $\boldsymbol{L} \times \ddot{\boldsymbol{r}}$.

$$\begin{aligned} \boldsymbol{L} \times \ddot{\boldsymbol{r}} &= (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \times \left(-\frac{\mu\boldsymbol{r}}{r^3}\right) && (4.79) \\ &= -\frac{\mu}{r^3}\boldsymbol{r} \times \dot{\boldsymbol{r}} \times \boldsymbol{r} && (4.80) \end{aligned}$$

The triple vector product is $\boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{B}\left(\boldsymbol{A} \cdot \boldsymbol{C}\right) - \boldsymbol{C}\left(\boldsymbol{A} \cdot \boldsymbol{B}\right)$, leading to

$$\boldsymbol{L} \times \ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\left[(\boldsymbol{r} \cdot \boldsymbol{r})\,\dot{\boldsymbol{r}} - (\boldsymbol{r} \cdot \dot{\boldsymbol{r}})\,\boldsymbol{r}\right] \tag{4.81}$$

Given $\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = r\dot{r}$, we have

$$\boldsymbol{L} \times \ddot{\boldsymbol{r}} = -\mu\left(\frac{\dot{\boldsymbol{r}}}{r} - \frac{\boldsymbol{r}\dot{r}}{r^2}\right) = \frac{d}{dt}\left(-\frac{\mu\boldsymbol{r}}{r}\right) = \frac{d}{dt}\left(-\mu\hat{\boldsymbol{r}}\right) \tag{4.82}$$

Yet, we can also say

$$\boldsymbol{L} \times \ddot{\boldsymbol{r}} = \frac{d}{dt} (\boldsymbol{L} \times \dot{\boldsymbol{r}}) \tag{4.83}$$

Equating Eq. (4.82) and Eq. (4.83),

$$\frac{d}{dt} (\boldsymbol{L} \times \dot{\boldsymbol{r}} + \mu \hat{\boldsymbol{r}}) = 0 \tag{4.84}$$

From which we conclude that

$$\boldsymbol{L} \times \dot{\boldsymbol{r}} + \mu \hat{\boldsymbol{r}} \equiv \text{const} \tag{4.85}$$

This is not the most transparent vector, but at least one property is immediate. The first term must lie in the plane of motion, because $\boldsymbol{L}$ is perpendicular to it and $\dot{\boldsymbol{r}}$ is on the plane. The second term is also in the planet of motion, so the conclude that the vector is in the plane of motion.

We can tell the direction of the vector by taking its dot product with the radius vector. Let us call this constant vector $\boldsymbol{A}$. Then by definition

$$\boldsymbol{r} \cdot \boldsymbol{A} = rA \cos\theta \tag{4.86}$$

Given

$$\boldsymbol{A} = \boldsymbol{L} \times \dot{\boldsymbol{r}} + \mu \hat{\boldsymbol{r}} \tag{4.87}$$

We can calculate

$$\boldsymbol{r} \cdot \boldsymbol{A} = \boldsymbol{r} \cdot (\boldsymbol{L} \times \dot{\boldsymbol{r}}) + \mu \boldsymbol{r} \cdot \hat{\boldsymbol{r}} \tag{4.88}$$

The first term is a scalar triple product. We can use the identities

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \tag{4.89}$$

to write

$$\boldsymbol{r} \cdot (\boldsymbol{L} \times \dot{\boldsymbol{r}}) = \boldsymbol{L} \cdot (\dot{\boldsymbol{r}} \times \boldsymbol{r}) \tag{4.90}$$

$$= -L^2 \tag{4.91}$$

where the last equality comes from the definition of the angular momentum. So

$$\boldsymbol{r} \cdot \boldsymbol{A} = -L^2 + \mu r \tag{4.92}$$

Equating Eq. (4.86) and Eq. (4.92) and solving for r

$$r = \frac{L^2/\mu}{1 - (A/\mu)\cos\theta} \tag{4.93}$$

Comparing with Newton's orbital solution we find that the vector has amplitude $A = -\mu e$, and is oriented such that the angle that it makes with the position vector $\boldsymbol{r}$ is $\theta = f$, i.e., the true anomaly. The conclusion is that the vector is parallel to the line of apsides, pointing to the direction of periastron. Based on these we can define the vector $\boldsymbol{e}$

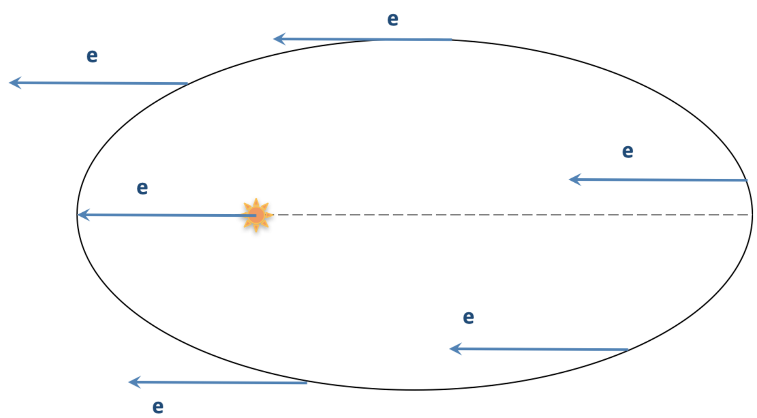

Figure 4.4: The eccentricity vector $\boldsymbol{e}$, or Runge-Lenz vector, is a constant of motion. It points towards the direction of periastron, $\boldsymbol{r} \cdot \boldsymbol{e} \propto \cos f$. If this vector is constant, then we conclude that periastron is always at the same position, i.e., not only the plane of the orbit is fixed, but the orbit itself does not change orientation.

$$\boxed{\boldsymbol{e} = -\frac{1}{\mu}\boldsymbol{L} \times \dot{\boldsymbol{r}} - \hat{\boldsymbol{r}} \equiv const} \quad (4.94)$$

called the *eccentricity vector*, or Runge-Lenz vector. This vector is a constant of motion, of amplitude equal to the orbital eccentricity, and ponting to the direction of pericenter.

One profound implication of the constancy of the eccentricity vector is that the direction of perihelion, and thus of the line of apsides, if fixed. The orbit does not change its spatial orientation.

4.4 Orbital parameters and physical parameters

Take the dot product of the eccentricity vector with itself

$$e^2 = \left(\mu^{-1}\boldsymbol{L} \times \dot{\boldsymbol{r}} + \hat{\boldsymbol{r}}\right) \cdot \left(\mu^{-1}\boldsymbol{L} \times \dot{\boldsymbol{r}} + \hat{\boldsymbol{r}}\right) \quad (4.95)$$

$$= \mu^{-2}\left(\boldsymbol{L} \times \dot{\boldsymbol{r}}\right) \cdot \left(\boldsymbol{L} \times \dot{\boldsymbol{r}}\right) + 1 + 2\mu^{-1}\left(\boldsymbol{L} \times \dot{\boldsymbol{r}}\right) \cdot \hat{\boldsymbol{r}} \quad (4.96)$$

The first term in the RHS is the square of the modulus of $\boldsymbol{L} \times \dot{\boldsymbol{r}}$. Because $\boldsymbol{L}$ and $\dot{\boldsymbol{r}}$ are perpendicular, $|\boldsymbol{L} \times \dot{\boldsymbol{r}}| = |\boldsymbol{L}||\dot{\boldsymbol{r}}| = Lv$, so

$$e^2 - 1 = \mu^{-2}L^2v^2 + 2\mu^{-1}\left(\boldsymbol{L} \times \dot{\boldsymbol{r}}\right) \cdot \hat{\boldsymbol{r}} \quad (4.97)$$

For the second term in the RHS we can again use the vector identities of the triple product to write

$$\hat{\boldsymbol{r}} \cdot (\boldsymbol{L} \times \dot{\boldsymbol{r}}) = \boldsymbol{L} \cdot (\dot{\boldsymbol{r}} \times \hat{\boldsymbol{r}}) = -L^2/r \tag{4.98}$$

So,

$$e^2 - 1 = \mu^{-2} L^2 v^2 - 2\mu^{-1} r^{-1} L^2 \tag{4.99}$$

$$= \frac{2L^2}{\mu^2}\left(\frac{v^2}{2} - \frac{\mu}{2}\right) \tag{4.100}$$

The quantity between parentheses in the RHS is the energy. We can then isolate the eccentricity to find

$$\boxed{e = \sqrt{1 + \frac{2EL^2}{\mu^2}}}. \tag{4.101}$$

Let us now use the definition of the semilatus rectum

$$p = \frac{L^2}{\mu} = a(1 - e^2) \tag{4.102}$$

Susbtituting Eq. (4.101), $1 - e^2 = -2EL^2/\mu^2$, we find an expression for the semi-major axis

$$\boxed{a = -\frac{\mu}{2E}} \tag{4.103}$$

On Eqs 4.101 and 4.103, the LHS is geometrical, the RHS is physical. These equations synthesize the combination of Kepler's laws, which are empirical, with Newtonian physics, explaining the causes of the motion. The semimajor axis depends on the energy alone. Eccentricity is set by a combination of the energy and the angular momentum.

4.4.1 Bound orbits

Considering again the definition of the semilatus rectum, and writing a in terms of energy

$$L^2 = -\frac{\mu^2(1 - e^2)}{2E} \tag{4.104}$$

we see that for constant angular momentum, if we increase the energy we need to increase the eccentricity. Increasing the energy without increasing the angular momentum can be done by adding a radial velocity.

Considering the energy

$$E = \frac{v^2}{2} - \frac{\mu}{r} \tag{4.105}$$

It is a combination of kinetic energy and potential energy. If the energy is negative, it means that the gravitational potential energy is larger than the kinetic energy, and the orbit is bound. Considering that the energy is negative, then Eq. (4.101) implies that the eccentricity is $0 \leq e < 1$, and the orbit is either circular ($e = 0$) or elliptic ($0 < e < 1$).

4.4.2 Unbound orbits

If, on the other hand, the energy is zero, the system has just enough energy to reach infinity with zero velocity. In this case, $e = 1$. What orbit has $e = 1$?

According to Eq. (4.103), the semimajor axis is infinite. According to Eq. (4.104), the angular momentum is indefinite. Let us identify the geometry of $e = 1$

$$r|_{e=1} = \frac{L^2/\mu}{1+\cos f} \tag{4.106}$$

at perihelion, $f = 0$, $r = L^2/\mu$, which is also the semilatus rectum. At aphelion, $f = \pi$, and $r \to \infty$. The orbit is not periodic, as expected for an unbound orbit.

We can understand the geometry of the unbound orbit by writing the orbit in terms of finite quantities: e, and p. Using a reference frame centered at the periastron, the planet position is

$$x = q + r\cos\beta \tag{4.107}$$
$$y = r\sin\beta \tag{4.108}$$

where $-q$ is the perihelion position in the heliocentric frame. Geometrically,

$$y^2 = r^2 - (x-q)^2 \tag{4.109}$$

We can consider the orbit r in terms of the true anomaly f, and since $f = \beta - \pi$ we can substitute $\cos f = -\cos\beta = -(x-q)/r$, writing the orbit as

$$r = p + e(x-q) \tag{4.110}$$

Thus in Cartesian coordinates

$$y^2 = p^2 + (e^2-1)(x-q)^2 + 2pe(x-q) \tag{4.111}$$

Considering $p = a(1-e^2)$ and $q = a(1-e)$, then we can substitute

$$p = q(1+e) \tag{4.112}$$

$$y^2 = p^2 + (e^2-1)\left(x^2 + \frac{p^2}{(1+e)^2} - \frac{2xp}{1+e}\right) + 2pex - \frac{2p^2e}{1+e} \tag{4.113}$$
$$= p^2 + (e^2-1)x^2 + \frac{p^2(e-1)}{(1+e)} - 2xp(e-1) + 2pex - \frac{2p^2e}{1+e} \tag{4.114}$$
$$= p^2 + (e^2-1)x^2 + \frac{p^2e}{(1+e)} - \frac{p^2}{(1+e)} + 2px - \frac{2p^2e}{1+e} \tag{4.115}$$

This 2nd, 3rd and last term sum up to p^2, so the equation simplifies to

$$y^2 = (e^2-1)x^2 + 2px \tag{4.116}$$

This equation is known as the *pencil* of the orbit. If $e = 1$,

$$y^2 = 2px \tag{4.117}$$

which is a *parabola*. The semilatus rectum is $p = 2q$.

If the orbit has no angular momentum, $L^2/\mu \equiv p = 0$, yielding then $y^2 = 0$ at all points, i.e., the orbit degenerates into a line crossing the Sun (periastron at $q = 0$).

If the energy is positive, then kinetic energy exceeds the gravitational potential energy, and the body reaches infinity with nonzero velocity. The eccentricity is $e > 1$. In this case, we can use the equation of the ellipse, centered in the bisecting point of the line joining the foci, and keeping the signs positive,

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1 \tag{4.118}$$

which is the equation of a *hyperbola*. The hyperbola has positive energy and eccentricity larger than 1. The equations derived for the ellipse seem to imply that the semimajor axis is negative. Yet, as a geometrical construct the semimajor axis and the semilatus rectum are positive. So, for the hyperbola

$$a = \frac{\mu}{2E} \tag{4.119}$$

and the semilatus rectum is

$$p = a(e^2 - 1) \tag{4.120}$$

The sun-centered orbit is thus

$$r = \frac{p}{1 + e\cos f} \tag{4.121}$$

with the different geometries defined by

Geometry	eccentricity e	semilatus rectum ($p = L^2/\mu$)	energy
circle	$e = 0$	$p = a$	$E = E_{\min}$
ellipse	$0 < e < 1$	$p = a(1 - e^2)$	$E_{\min} < E < 0$
parabola	$e = 1$	$p = 2q$	$E = 0$
hyperbola	$e > 1$	$p = a(e^2 - 1)$	$E > 0$

The minimum energy is that of the circular orbit, with constant velocity given by the centrifugal force

$$v^2 = \frac{\mu}{a} \tag{4.122}$$

So

$$E_{\min} = -\frac{\mu}{2a} \tag{4.123}$$

The parabolic orbit has the velocity to escape with zero velocity at infinity. This defines the escape velocity

$$v_e^2 = \frac{2\mu}{r}, \tag{4.124}$$

which is the velocity of the parabolic orbit.

4.4.3 Effective potential

Considering the energy and angular momentum

$$E = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{\mu}{r} \tag{4.125}$$
$$L = r^2\dot{\phi} \tag{4.126}$$

Substituting L into E,

$$\frac{\dot{r}^2}{2} = E - \frac{L^2}{2r^2} + \frac{\mu}{r} \tag{4.127}$$

we can write this expression as

$$K = E - U_{\rm eff} \tag{4.128}$$

With K the kinetic energy, and $U_{\rm eff}$ an effective potential

$$\boxed{U_{\rm eff} = \frac{L^2}{2r^2} - \frac{\mu}{r}} \tag{4.129}$$

This potential is sketched in Fig. 4.6 for $L^2/2 = A = 1$ and $\mu = B = 2$. The first term is a centrifugal term, dubbed the centrifugal barrier (red dashed curve in Fig. 4.6). The second term is the potential well (blue dashed curve in Fig. 4.6). The effective potential is the black solid line. If $r \to 0$, then the centrifugal barrier dominates. If $r \to \infty$, then the potential term dominates.

The effective potential has a well-defined minimum. The very minimum of energy allows only one possible value for r, corresponding to the circular orbit. As the energy increases, a range of r becomes allowed. As long as $E < 0$, the orbit is closed, limited between two extrema. This corresponding to an elliptical orbit (Fig. 4.7). The range are $r_{\rm min} = q = a(1-e)$, the periastron, and $r_{\rm max} = Q = a(1+e)$, the apoastron. The circular and elliptical orbits are said to be *bound*.

The semimajor axis increases with $E = -B/2a$, traced as the yellow solid line in Fig. 4.8. As E approaches zero, the semimajor axis approaches infinity, and so does the apoastron. When $E \geq 0$, the orbiter is able to reach infinity with either zero velocity ($E = 0$) or nonzero velocity ($E > 0$), corresponding to parabolic or hyperbolic, i.e., open orbits. These orbits are said to be *unbound*.

4.5 Orbit determination observationally

The orbit is in a plane perpendicular to the angular momentum vector $\boldsymbol{L}$. Aligning the z-axis with $\boldsymbol{L}$, the value of $\boldsymbol{r} = (x, y)$ and $\boldsymbol{v} = (\dot{x}, \dot{y})$ at any given time define a unique orbit given by a and e. The extra variables define the argument of perihelion ω and the true anomaly f.

If we consider other planes so that $\boldsymbol{L}$ is not aligned with $\hat{z}$, then the extra variables z and $\dot{z}$ yield the inclination i and the longitude of the ascending node Ω.

To find the orbital elements we must first change plane from the orbital plane of the body to the reference frame we choose. Usually for Solar System studies we use heliocentric coordinates (X, Y, Z) where the reference plane $Z = 0$ is the ecliptic, and the reference line for the X coordinate is the direction of the vernal equinox. We thus need to transform from (x, y, z) to (X, Y, Z). We rotate first around the z axis by an angle *omega* so that x coincides with the line of nodes, then a rotating about the x axis by i so that the planes are coincident, and a rotation about the z axis by Ω to align with the vernal equinox. These are three rotation matrixes

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P_3 P_2 P_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.130}$$

with $P_1 = R_z(\omega)$, $P_2 = R_x(i)$, and $P_3 = R_z(\Omega)$. So,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P_3 P_2 P_1 \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix} = r \begin{pmatrix} \cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i \\ \sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i \\ \sin(\omega+f)\sin i \end{pmatrix} \tag{4.131}$$

Having $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, we can write

$$R^2 = X^2 + Y^2 + Z^2 \tag{4.132}$$
$$V^2 = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \tag{4.133}$$
$$\boldsymbol{L} = (Y\dot{Z} - Z\dot{Y}, Z\dot{X} - X\dot{Z}, X\dot{Y} - Y\dot{X}) \tag{4.134}$$

The rate of change of the radius vector is

$$\dot{R}^2 = V^2 - \frac{L^2}{R^2} \tag{4.135}$$

The sign of $\dot{R} = \pm\sqrt{\dot{R}^2}$ is the sign of

$$\boldsymbol{R}\cdot\dot{\boldsymbol{R}} = X\dot{X} + Y\dot{Y} + Z\dot{Z} \tag{4.136}$$

Because the energy of the orbit is constant

$$E = -\frac{\mu}{2a} \tag{4.137}$$

The velocity can be written in terms of the radius and semimajor axis

$$V^2 = \mu\left(\frac{2}{R} - \frac{1}{a}\right) \tag{4.138}$$

We can invert this equation to find a

$$\boxed{a = \left(\frac{2}{R} - \frac{V^2}{\mu}\right)^{-1}} \tag{4.139}$$

And also given $\boldsymbol{L}$, we can find the eccentricity

$$\boxed{e = \sqrt{1 - \frac{L^2}{a\mu}}} \tag{4.140}$$

Taking the projection of $L = (L_x, L_y, L_z)$ onto the three axes

$$\begin{aligned} L_x &= \boldsymbol{L} \cdot \hat{\boldsymbol{x}} = L \sin i \cos \Omega & (4.141)\\ L_y &= \boldsymbol{L} \cdot \hat{\boldsymbol{y}} = L \sin i \sin \Omega & (4.142)\\ L_z &= \boldsymbol{L} \cdot \hat{\boldsymbol{z}} = L \cos i & (4.143)\\ & & (4.144) \end{aligned}$$

The inclination is thus

$$\boxed{i = \cos^{-1}\left(\frac{L_z}{L}\right)} \qquad (4.145)$$

The longitude of the ascending node is found by

$$\boxed{\sin \Omega = \frac{L_x}{L \sin i} \quad \text{and} \quad \cos \Omega = \frac{L_y}{L \sin i}} \qquad (4.146)$$

$\omega + f$ is found by the expressions of X/R and Z/R

$$\begin{aligned} \sin(\omega + f) &= \frac{Z}{R \sin i} & (4.147)\\ \cos(\omega + f) &= \frac{1}{\cos \Omega}\left(\frac{X}{R} + \sin \Omega \sin(\omega + f) \cos i\right) & (4.148) \end{aligned}$$

Finally the true anomaly (and thus ω) are found from the equation of the orbit and its derivative

$$\boxed{\cos f = \frac{1}{e}\left[\frac{a(1-e^2)}{R} - 1\right] \quad \text{and} \quad \sin f = \frac{a(1-e^2)}{Le}\dot{R}} \qquad (4.149)$$

Problems

1. The *Kepler* mission was designed to find Earth-like planets around other stars.

   (a) On the graph shown in Fig. 4.9, sketch a line indicating where planets with a similar orbital period to the Earth would be.

   (b) Where would planets with 27-year orbits be on this plot?

   Show your work and/or explain your reasoning.

2. Fig. 4.10 shows the orbital period and semi-major axis of the six planets discovered by the *Kepler* Mission around a star named Kepler 11. Are the data consistent with Kepler's 3rd law? Be quantitative. Use the data in the graph to find the mass of Kepler 11 in kilograms and in solar mass units. Clearly explain your method. Assume that the planets are much less massive than the star.

3. As a satellite in circular orbit around the Earth loses energy due to friction with the thin upper atmosphere, does the satellite spiral inward or spiral outward? Does its orbital speed increase or decrease? Explain your reasoning carefully using concepts of energy.

	Radius of Moon	Distance from Planet
Moon A	100 km	100,000 km
Moon B	1,000 km	200,000 km
Moon C	1,000 km	300,000 km

4. Based on Kepler's 3rd law, what is the mass of the Sun?

5. Weighting the planets.

   (a) Phobos orbit Mars with semimajor axis of 9376 km and period 8 hours. What is the mass of Mars? Give your answer in Earth masses and Solar masses.

   (b) Ganymede orbits Jupiter with semimajor axis 1 070 400 km and orbital period 7 days. What is the mass of Jupiter? Give your answer in Earth masses and Solar masses.

   (c) Titan orbits Saturn with semimajor axis 1 221 870 km and period of 16 days. What is the mass of Saturn? Give your answer in Earth masses and Solar masses.

   (d) Titania orbits Uranus with semimajor axis 435 910 km and period 9 days. What is the mass of Uranus. Give your answer in Earth masses and Solar masses.

   (e) Triton orbits Neptune with semimajor axis 354 759 km and period 6 days. What is the mass of Neptune? Give your answer in Earth masses and Solar masses.

   (f) Mercury and Venus do not have satellites. How do we know their masses?

6. A spacecraft arrives and enters a circular, synchronous orbit around Mars (with its orbital period equal to Mars' rotational period, 24h 37min). What will be its orbital velocity?

7. Consider the following system of moons around a giant planet. Assume all moons have the same mass and are in circular orbits. For all parts below, explain your method and/or reasoning clearly.

   (a) Rank the moons in order from fastest to slowest orbital speed

   (b) Rank the moons in order of highest to lowest orbital energy

8. (a) Comet 46P/Wirtanen, currently visible in the sky, has semimajor axis a = 3.09 AU and eccentricity e = 0.66. What are its perihelion and aphelion distances?

   (b) A generic comet moves in a parabolic orbit. Find its velocity when its distance from the Sun is 100 AU.

9. The two-body problem describes the motion of two bodies around their common center of mass, or barycenter. In the Solar Sytem, the mass ratio of the bodies involved is such that the center of mass of most systems lie within the radius of the primary. One notable exception is the Pluto-Charon system, where the

barycenter is noticeably outside Pluto. Fig. 4.11 shows a sketch of the Pluto-Charon system. The distance r_1 from Pluto's center to the barycenter is 2110 km. The distance r_2 from the barycenter to Charon's center is 17 536 km. The orbital period of both Pluto and Charon around the barycenter is 6.38 days.

(a) What is the sum of the masses of Pluto and Charon, in kg? For comparison, the mass of the Moon is 7.3×10^{22} kg.

(b) Knowing that the location of the center of mass of a system of N masses m_i located at positions $\boldsymbol{r}_i$ is given by

$$\boldsymbol{r}_{\rm CM} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{\sum_{i=1}^{N} m_i} \tag{4.150}$$

What are the individual masses of Pluto and Charon, in kg?

10. (a) Identify in Fig. 4.12 the angles A, B, C, and D as the orbital elements: argument of periastron, longitude of ascending node, true anomaly and inclination. The vernal equinox is identified by the letter γ.

(b) Identify the respective orbital elements in the drawing of Fig. 4.13, and apply the Gauss groups of spherical trigonometry to find the following expressions for the ecliptic latitude λ and longitude β of a planet in terms of the orbital elements

$$\sin\beta = \sin i \sin(\omega + f) \tag{4.151}$$

$$\tan(\lambda - \Omega) = \cos i \tan(\omega + f) \tag{4.152}$$

where i is the inclination, ω is the argument of perihelion, f the true anomaly, and Ω the longitude of the ascending node.

(c) Jupiter's orbital eccentricity is $e = 0.05$. Find the mean, eccentric, and true anomaly of Jupiter one quarter of an orbit after perihelion. Use accuracy to first order in eccentricity to solve Kepler's equation.

(d) Jupiter's semimajor axis is $a = 5.2$ AU. What is Jupiter's distance to the Sun a quarter of an orbit after perihelion?

(e) The remainder of Jupiter's orbital elements are $i = 1.3°$, $\Omega = 100.5°$, $\omega = 274.2°$. What are Jupiter's ecliptic latitude and longitude a quarter of an orbit after perihelion?

11. A test particle approaches a planet of mass M and radius R from infinity with speed v_∞ and impact parameter b (Fig. 4.14).

(a) Use the particle's energy and angular momentum with respect to the planet to derive expressions for the semimajor axis and eccentricity of the hyperbolic orbit followed by the test particle about the planet.

(b) Show that the periastron distance is

$$q = \frac{GM}{v_\infty^2}\left[\sqrt{1 + \left(\frac{b v_\infty^2}{GM}\right)^2} - 1\right] \tag{4.153}$$

(c) Show that the eccentricity may be written

$$e = 1 + \frac{2v_\infty^2}{v_0^2} \tag{4.154}$$

where v_0 is the escape velocity at periastron. The escape velocity is the velocity of the parabolic orbit.

(d) For a hyperbolic orbit, as $r \to \infty$ the body's trajectory approaches an asymptote. Given the equation of the orbit

$$r = \frac{a(1-e^2)}{1 + e\cos f} \tag{4.155}$$

the asymptotic true anomaly corresponds to $\cos f_\infty = -e^{-1}$. Show that the deflection of the test particle's orbit after it leaves the vicinity of the planet, ψ, is given by $\sin(\psi/2) = e^{-1}$.

(e) Given that q must be greater than R to avoid a physical collision, calculate the maximum deflection angle

   i. for a spacecraft skimming Jupiter, with $v_\infty = 10\,\mathrm{km\,s^{-1}}$. Jupiter's mass is 2×10^{27} kg, and its radius is 70 000 km.

   ii. the *Cassini* orbiter skimming Saturn's large moon Titan, at $v_\infty = 5\,\mathrm{km\,s^{-1}}$.

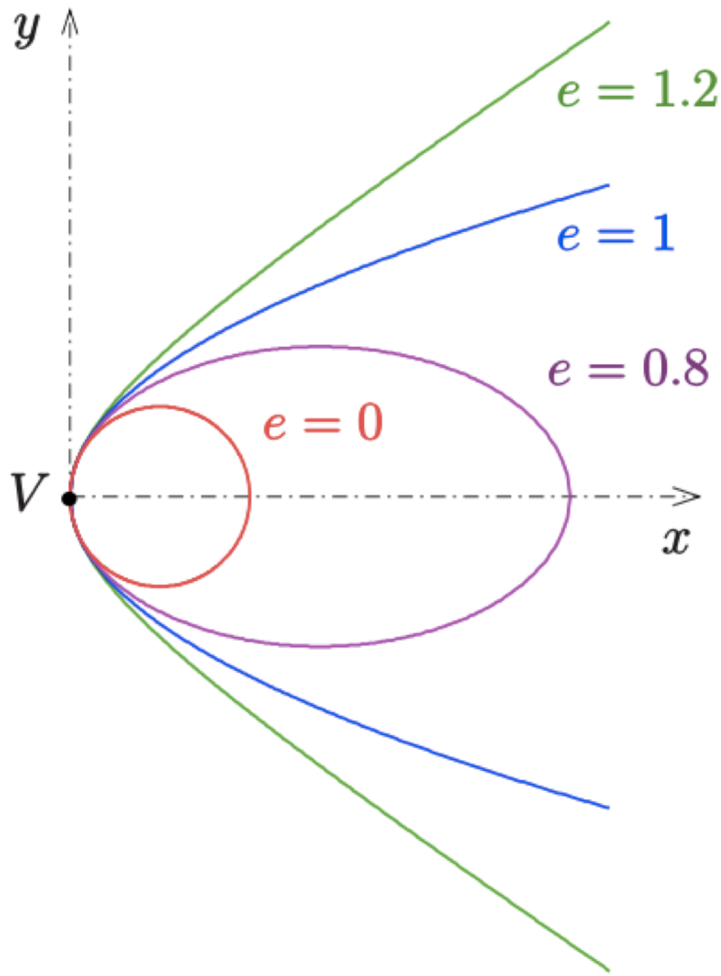

Figure 4.5: Depending on the eccentricity, the orbit can be a circle ($e = 0$, red), an ellipse ($0 < e < 1$, purple), a parabola ($e = 1$, blue), or a hyperbola ($e > 1$, green).

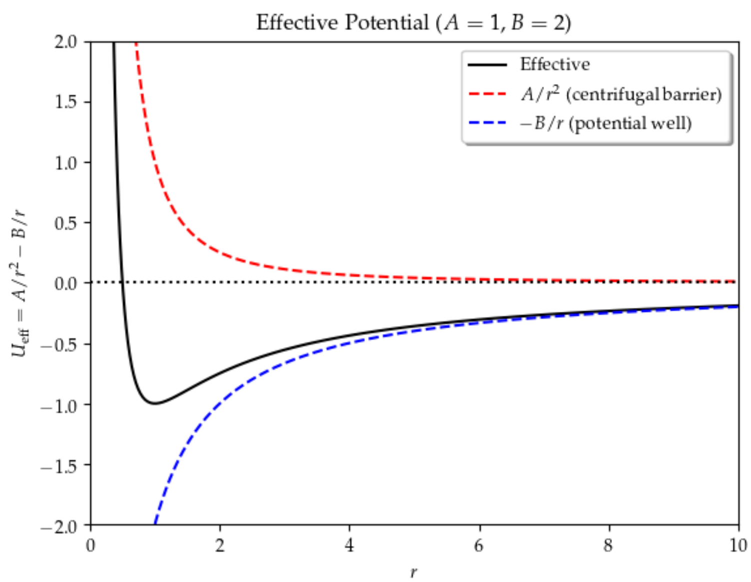

Figure 4.6: The combination of the gravitational potential well with the centrifugal force makes the planet feel an effective potential. At close separations the centrifugal barrier dominates. At large distances gravity dominates. A well-defined well exists; the energy minimum corresponds to the circular orbit.

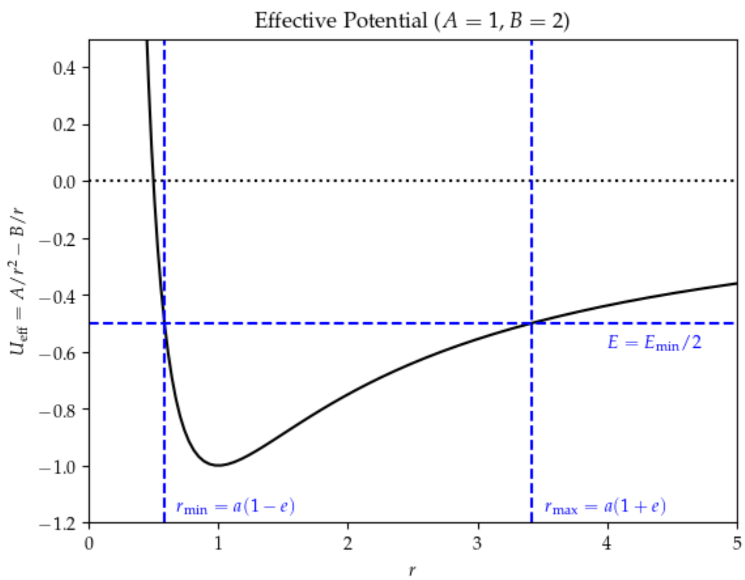

Figure 4.7: At higher energies, but still negative, the motion is bound between two extrema. These correspond to the elliptical orbit.

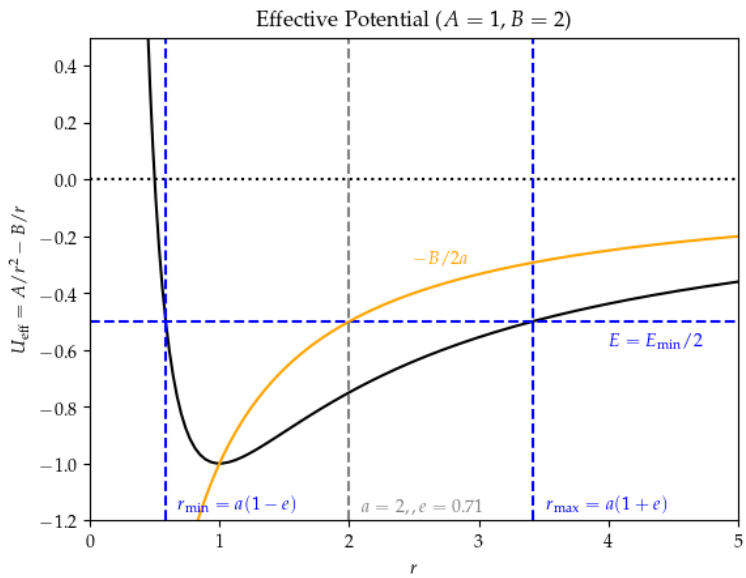

Figure 4.8: The semimajor axis increases with energy. When $E = 0$ the orbit becomes open (unbound), as the object is able to reach infinity and does not return.

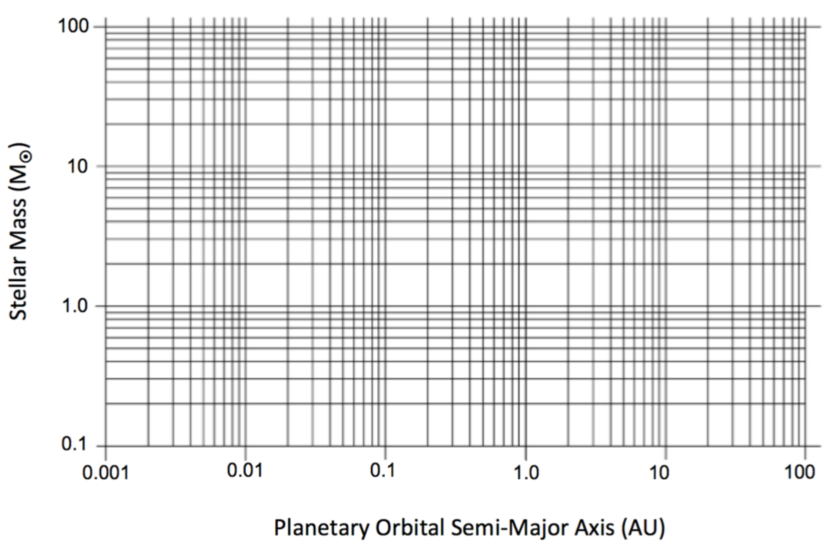

Figure 4.9: .

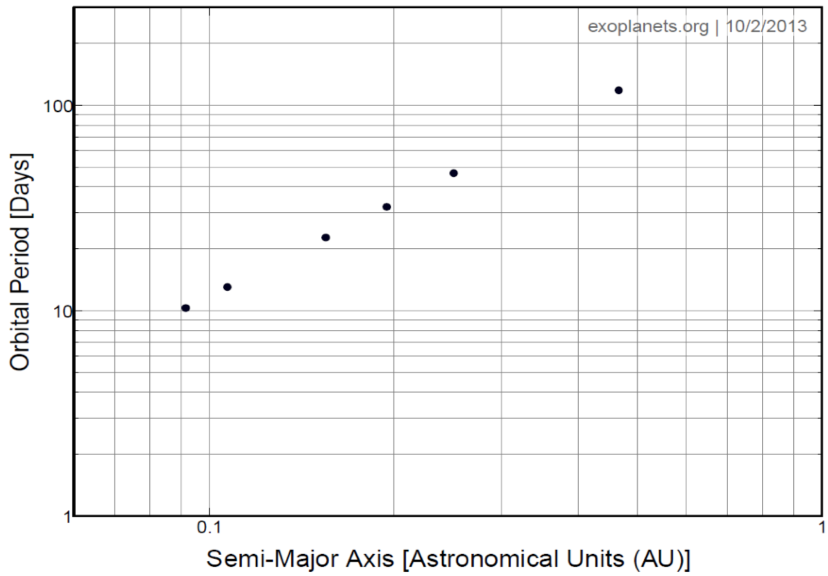

Figure 4.10: .

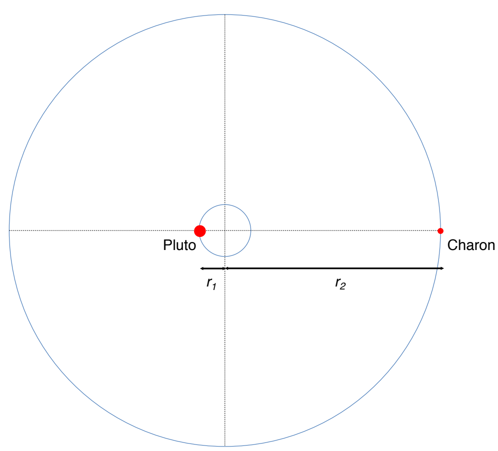

Figure 4.11: The Pluto-Charon system.

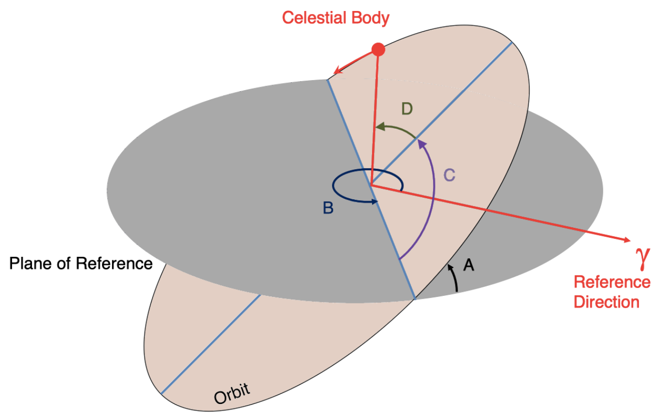

Figure 4.12: Orbital Elements. q marks perihelion and γ the vernal equinox.

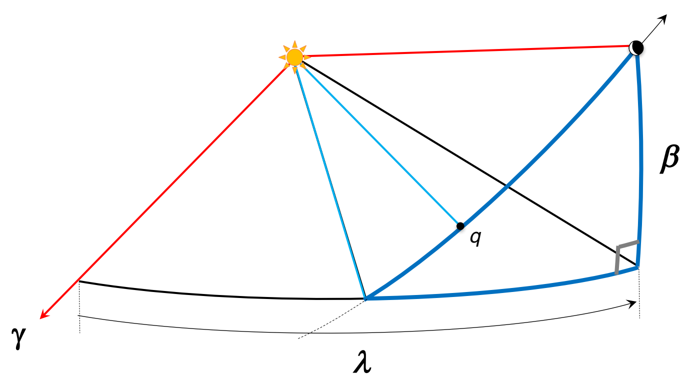

Figure 4.13: Orbital elements and coordinates.

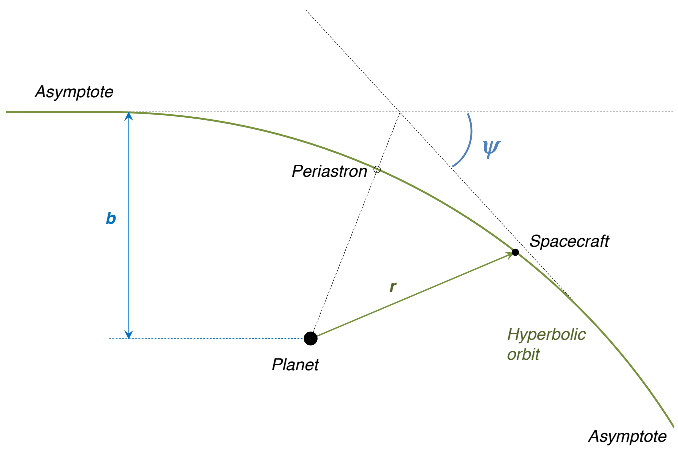

Figure 4.14: Approaching the planet with impact parameter b, a body will be deflected by an angle ψ.