

## Chapter 5

# Radiative Transfer

### 5.1 Principles of Radiative Transfer

The theory of radiative transfer is a macroscopic description of radiation fields. We will at first have to ask you to unwire one of the results of modern physics, that light consists of quanta. Instead of photons, we will think in terms of *beams*. This is because Radiative Transfer as a theory precedes quantum physics and, after all, for quantities of energy much larger than that of a single photon, a macroscopic description in terms of rays of light carrying energy in straight lines is more appropriate.

#### 5.1.1 Flux

We are concerned with the energy in these rays and how they interact with matter. One of the most primitive concepts is that of the energy *flux*, an intuitive measurement of energy flow. Imagine that you have a detector of area  $A$ , and you collect radiation with it, for a time interval  $t$ . If the source is constant, the amount of energy you collect will be proportional to  $A$  and to  $t$ . To be able to write a proportionality, we consider that even if the energy source is varying, during an infinitesimal time interval  $dt$  the source can be considered constant in time. Likewise, if the area is too large, different areas of the detector can also be detecting different amounts of energy, so we also consider the area to be infinitesimal. In this infinitesimal patch  $dA$ , the energy is spatially uniform. We define thus the idea of energy flowing through an area over time

$$dE \propto dA dt \quad (5.1)$$

and we call the proportionality factor the *flux*, giving it the symbol  $F$

$$dE \equiv F dA dt \quad (5.2)$$

This equation<sup>1</sup> is in fact still missing a physical ingredient that the energy depends on. If your detector is sensitive only in, say, X-rays, but the radiation shining on it is, say, microwaves, then no energy will be measured. We conclude that we must also define the *frequency range* in which we are measuring the radiation. The *specific flux*  $F_\nu$  is then the net energy flowing through per unit area per unit time in a given frequency

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<sup>1</sup>A note on notation: as there are two infinitesimals in the right-hand side, the left-hand side should not be  $dE$  but  $d^2E$ . In practice, radiative transfer theory tends to not use this more rigorous notation, and exponents in infinitesimals are dropped.

range. We define all these intervals to be infinitesimal so that the energy is constant in space, time, and frequency ranges

$$dE_\nu = F_\nu dA dt d\nu \quad (5.3)$$

or

$$F_\nu \equiv \frac{dE_\nu}{dA dt d\nu} \quad (5.4)$$

The subscript  $\nu$  in  $F_\nu$  and  $dE_\nu$  denotes that these quantities are *monochromatic*, i.e., measured in a single frequency. In practice, we cannot measure monochromatic fluxes. We always measure a finite range of frequencies (a waveband). So, when we measure flux in a given waveband, from  $\nu_1$  to  $\nu_2$ , what we are measuring is

$$F = \int_{\nu_1}^{\nu_2} F_\nu d\nu. \quad (5.5)$$

The left-hand side is what is measured. The integrand in the right-hand side is mathematical abstraction. The quantity  $F_\nu$  is an underlying function, an idealized monochromatic flux. So that the units of Eq. (5.5) come out correctly,  $F_\nu$  must be defined per hertz. The unit<sup>2</sup> of monochromatic flux  $F_\nu$  is thus

$$[F_\nu] = \text{energy (time)}^{-1} (\text{area})^{-1} (\text{frequency})^{-1} \quad (5.6)$$

$$= \text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1} \quad (5.7)$$

whereas the unit of flux  $F$  is

$$[F] = \text{energy (time)}^{-1} (\text{area})^{-1} \quad (5.8)$$

$$= \text{erg s}^{-1} \text{cm}^{-2}. \quad (5.9)$$

We can also define the *bolometric flux*, which is simply the total flux, integrated in all frequencies.

$$F_{\text{bol}} = \int_0^\infty F_\nu d\nu \quad (5.10)$$

### 5.1.1.1 The flux follows an inverse square law

Considering an isolated star, if we put spherical surfaces  $s$  and  $S$  of radius  $r$  and  $R$  around it (Fig. 5.1, left), by conservation of energy the total energy passing through  $s$  and  $S$  must be the same. Thus,

$$F(r) 4\pi r^2 = F(R) 4\pi R^2 \quad (5.11)$$

or

<sup>2</sup>In astronomy we often use cgs units instead of SI, which you may be more used to. In cgs, the unit of length is *centimeter* (1 cm = 10<sup>-2</sup> m), the unit of mass is *gram* (1 g = 10<sup>-3</sup> kg), and the unit of time is *second*, as in SI. The unit of energy, the *erg*, is built from these, being erg ≡ g cm<sup>2</sup>/s<sup>2</sup>. The equivalence between that and SI is 1 erg = 10<sup>-7</sup> J. A list of units and constants in cgs is shown in the end of this chapter

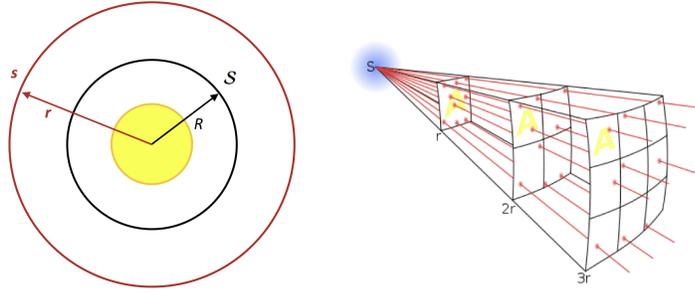


Figure 5.1: *Left*: The energy leaving the stellar surface is the same energy radiated through the surfaces  $S$  and  $s$  of radii  $R$  and  $r$ , respectively. *Right*: At greater radii, the same amount of energy disperses through a larger area. The flux (energy per area) thus decreases at the same rate that the area increases, with the square of the distance.

$$F(r) = F(R) \left( \frac{R}{r} \right)^2 \quad (5.12)$$

If we consider  $R$  instead as the radius of the star and  $r$  an arbitrary location away from the stellar surface, Eq. (5.12) says that the flux falls with the square of the distance. This is merely a statement of conservation of energy (Fig. 5.1, right).

## 5.2 Magnitudes

When we measure how bright a star is, we are measuring the energy flux. Astronomers use a historic scale to measure stellar brightness, which was defined in Ancient Greece (by either Hipparchus or Ptolemy, references vary), and called *magnitudes*. In Ptolemy's *Almagest*, the brightest stars are assigned first magnitude, and the faintest ones sixth magnitude, defining thus a ranking system of six categories. Being a system devised by naked eye measurements, the scale is nearly logarithmic. In linear scale, the difference in flux between a star of 1st and 6th magnitude is approximately 100. In the 19th century, the astronomer Norman Pogson suggested to make this a standard, i.e., to fix a difference of 5 magnitudes as a factor 100 in flux

$$m_1 - m_2 = 5 \iff \frac{F_2}{F_1} = 100 \quad (5.13)$$

Therefore, the magnitude scales with the flux according to

$$m = -2.5 \log F + C \quad (5.14)$$

where  $C$  is a constant. The constant is arbitrarily chosen so that the magnitude of the star Vega is zero.

### 5.2.1 Apparent and Absolute magnitudes

Because the flux depends on distance  $d$  following an inverse square law

$$F \propto \frac{1}{d^2}, \quad (5.15)$$

the magnitudes we measure are also a function of distance. For this reason, we call them *apparent magnitudes*.

We could define *absolute* magnitudes, independent of distances, that would reflect a star's true brightness. For that we need a standard distance  $D$ , that was arbitrarily defined as  $D = 10$  pc. The absolute magnitude  $M$  is thus

$$M = -2.5 \log [F(10pc)] + C \quad (5.16)$$

Notice that the difference between apparent and absolute magnitude

$$m - M = -2.5 \log \left[ \frac{F(d)}{F(D)} \right] \quad (5.17)$$

is a quantity that depends only on the distance. For this reason,  $m - M$  is also called the *distance modulus*. Because the flux follows an inverse square law, the distance modulus can be written as

$$m - M = 5 \log \left( \frac{d}{10pc} \right) \quad (5.18)$$

The distance, in parsecs, is then given by

$$d(\text{pc}) = 10^{0.2(m-M)+1} \quad (5.19)$$

It is also useful to express this as

$$M = m + 5 + 5 \log \pi'' \quad (5.20)$$

where  $\pi = 1/d(\text{pc})$ , measured in arcseconds, is the parallax angle (do not confuse it with the circle constant!). The equation above depends only on measurable quantities and is useful to have in handy when observing.

## 5.3 Intensity

The flux is a measurement of all rays that pass through a given area. A more fundamental description of the radiation field should consider the energy coming from *individual* rays (Fig. 5.2). If we were dealing with quantum mechanics, we would mean a single photon. However, as said, the theory of radiative transfer is a classical (or semi-classical) theory, describing radiation fields in a macroscopic way. We wish then to consider the energy in a single ray of radiation.

Yet, a single ray in practice cannot be isolated. What we measure is a set of rays, i.e., a *beam*. We can construct a beam considering a set of rays that differ slightly from the given ray. For that, construct an area  $S$  normal to the direction of the ray, and consider all rays that pass through  $S$  whose direction is within a solid angle  $\omega$  of the given ray (Fig. 5.3).

This beam will contribute an amount of energy at the detector related to the size of the solid angle. However, the radiation field may be non-uniform inside an extended

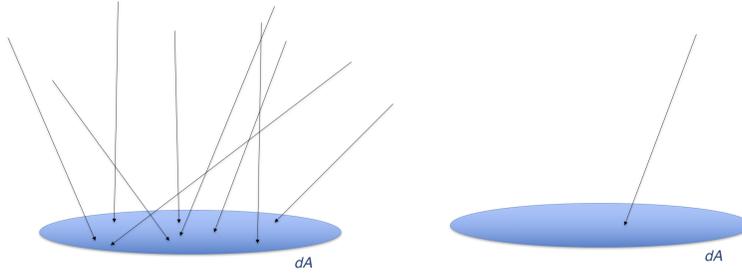


Figure 5.2: With a detector of area  $dA$  we measure flux, which is the sum of the energies of all rays crossing the area per unit time (left). A more fundamental quantity would be related to the energy of a single ray (right), if we could isolate it.

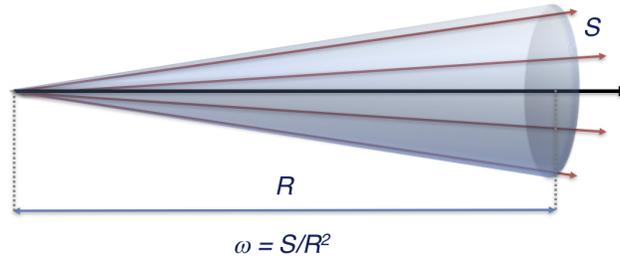


Figure 5.3: The solid angle  $\omega$  encompasses all rays that cross an area  $S$  normal to a given ray (solid black line).

beam. Thus, we consider an infinitesimal solid angle  $d\omega$ , within which we can consider the illumination to be uniform. Considered an infinitesimal beam, we can write the proportionality

$$dE \propto dA \cdot d\omega dt. \quad (5.21)$$

Notice that in this equation,  $dA$  is the area of the detector, and  $d\omega$  the solid angle of the beam. They are both vectors, pointing to their respective normals,  $\hat{n}$  and  $\hat{n}'$

$$d\mathbf{A} = \hat{n} dA \quad (5.22)$$

$$d\boldsymbol{\omega} = \hat{n}' d\omega \quad (5.23)$$

The dot product in Eq. (5.21) is intuitive: a ray that is grazing the detector (i.e., perpendicular to the normal to the surface of the detector) is not detected. If we define  $\theta$  being the plane angle between  $\hat{n}$  and  $\hat{n}'$ ,

$$dE \propto \hat{n} \cdot \hat{n}' dA d\omega dt \quad (5.24)$$

$$\propto \cos \theta dA d\omega dt \quad (5.25)$$

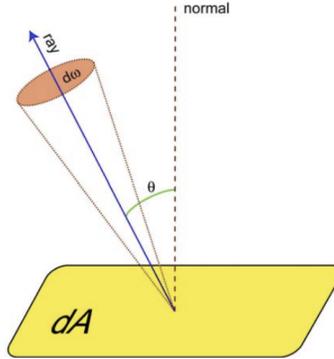


Figure 5.4: In the case of emitted radiation the area  $dA$  is an area element on the surface of a source, that emits radiation into a beam of directions encompassed by the solid angle  $d\omega$ .

We define the proportionality as the *intensity*. As was done for the flux, we can also define the specific intensity as the intensity in a frequency range

$$dE_\nu = I_\nu dA \cdot d\omega dt dv \quad (5.26)$$

or

$$I_\nu \equiv \frac{dE_\nu}{dA \cdot d\omega dt dv} \quad (5.27)$$

$$\equiv \frac{dE_\nu}{dA d\omega \cos \theta dt dv} \quad (5.28)$$

The intensity has dimension of

$$[I_\nu] = \text{energy (time)}^{-1} (\text{area})^{-1} (\text{solid angle})^{-1} (\text{frequency})^{-1} \quad (5.29)$$

$$= \text{erg s}^{-1} \text{cm}^{-2} \text{ster}^{-1} \text{Hz}^{-1} \quad (5.30)$$

Although we considered the intensity as measured from a direction  $d\omega$  into a detector of area  $dA$ , we can also think of emitted intensity. In this case, the geometry is as in Fig. 5.4: an area  $dA$  on the surface of a source emits radiation into a direction  $d\omega$ .

### 5.3.1 Constancy of intensity along a ray

When we take the limit  $\Delta\omega \rightarrow 0$ , the beam stops spreading out, approaching a single ray. Due to energy conservation, the energy is constant along a ray, and thus the intensity does not depend on distance. To prove this, consider two points,  $P_1$  and  $P_2$ , along a ray, separated by a distance  $r$  (Fig. 5.5). Consider the energy that leaves  $P_1$  and reaches  $P_2$ . These locations are points, so they do not fit our macroscopic definitions. We construct an area around  $P_1$  that will emit radiation, and an area around  $P_2$ , that will detect it. These areas are  $dA_1$  and  $dA_2$ . Consider now the solid angle  $d\omega_1$  that goes

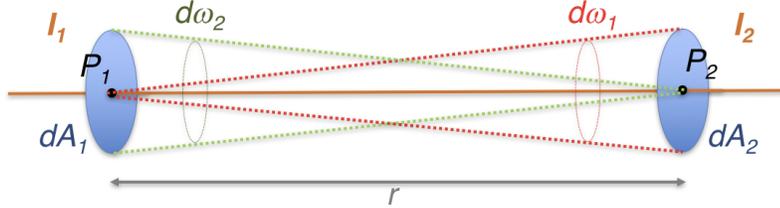


Figure 5.5: Constancy of intensity along a way. Radiation with Intensity  $I_1$  is emitted from  $P_1$ . What is the intensity  $I_2$  measured at  $P_2$ ? The energy emitted from the area  $dA_1$  into the cone  $d\omega_1$  is  $dE_1 = I_1 dA_1 d\omega_1 dt$ . The energy detected by area  $dA_2$  is  $dE_2 = I_2 dA_2 d\omega_2 dt$ . The solid angles are  $d\omega_1 = dA_2/r^2$  and  $d\omega_2 = dA_1/r^2$ . By energy conservation,  $I_1 = I_2$ . The intensity does not dilute with distance.

from  $P_1$  to  $dA_2$ . The energy flowing from the area  $dA_1$  around  $P_1$  into the solid angle that crosses  $dA_2$  is

$$dE_1 = I_1 dA_1 d\omega_1 dt. \quad (5.31)$$

From the point of view of  $P_2$ , it is collecting radiation with an area  $dA_2$ . The emitting source, of area  $dA_1$ , is seen from  $P_2$  at a distance  $r$ , forming a solid angle  $d\omega_2$ .  $P_2$  measures an amount of energy

$$dE_2 = I_2 dA_2 d\omega_2 dt \quad (5.32)$$

Because energy is conserved,  $dE_1 = dE_2$ , and we have

$$I_1 dA_1 d\omega_1 = I_2 dA_2 d\omega_2 \quad (5.33)$$

Substituting the definitions of the solid angles  $d\omega_1 = dA_2/r^2$  and  $d\omega_2 = dA_1/r^2$ , we conclude that  $I_1 = I_2$ . The intensities at  $P_1$  and  $P_2$  are *equal*. The reason is simple: as we make the solid angle  $d\omega$  infinitesimal, we can in principle make the beam so narrow that only a single photon can fit in its width. Because a photon does not spread its energy as it travels, the energy in the beam does not dilute with distance.

Notice that the fact that intensity does not depend on distance means that the surface of the Sun is as bright seen from Earth as it is seen from Neptune. This may seem counterintuitive at first, until we realize that a white wall does not get brighter as we approach it. Intensity is an intrinsic property of resolved sources of radiation.

### 5.3.2 Mean intensity

Although intensity depends on solid angle, we can define an averaged intensity that removes the dependency on direction. This directional average is also called the *mean intensity*, and defined as

$$J_\nu = \frac{\oint I_\nu d\omega}{\oint d\omega} \quad (5.34)$$

$$= \frac{1}{4\pi} \oint I_\nu d\omega \quad (5.35)$$

We can show that this quantity is related to the energy density of the radiation field. Consider the energy per unit volume per unit solid angle in a frequency range.

$$dE_\nu = u_{\nu,\omega} dV d\nu d\omega \quad (5.36)$$

The volume defined by a light beam is  $c dt dA$ , where  $c dt$  is the distance light travels in a unit time  $dt$ . So,

$$dE_\nu = u_{\nu,\omega} c dA dt d\nu d\omega = I_\nu dA dt d\nu d\omega \quad (5.37)$$

Comparing both identities, we find

$$u_{\nu,\omega} = I_\nu/c. \quad (5.38)$$

Integrating it in solid angle,

$$u_\nu = \frac{1}{c} \oint I_\nu d\omega \quad (5.39)$$

$$= \frac{4\pi}{c} J_\nu. \quad (5.40)$$

### 5.3.3 Relationship between intensity and flux

Flux is the net energy that passes per unit time through a unit area in a frequency range, as defined by Eq. (5.4). Comparing the definition of flux with the definition of intensity (Eq. 5.28) we conclude that

$$F_\nu = \oint I_\nu \hat{n} \cdot \hat{n}' d\omega = \oint I_\nu \cos \theta d\omega \quad (5.41)$$

i.e., the flux is a directional integral of the intensity. For an isotropic field, where  $I_\nu$  does not depend on direction, and we only see one hemisphere

$$F_\nu = \oint I_\nu \cos \theta d\omega = I_\nu \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \pi I_\nu \quad (5.42)$$

The flux is integrated in solid angle, so the directional information is eliminated. The intensity can only be defined for extended objects, angularly resolved, because we need to specify the interval of solid angles. Flux instead can be defined for any source, including point sources. Therefore, there is a transition between measuring intensity and measuring flux as we distance an observer from a source (Exercise 20).

Notice that if the angle  $\theta$  is small, as it usually is for astronomical sources, then  $\cos \theta \approx 1$  and

$$F \equiv \int I \cos \theta d\omega \approx I\omega \quad (5.43)$$

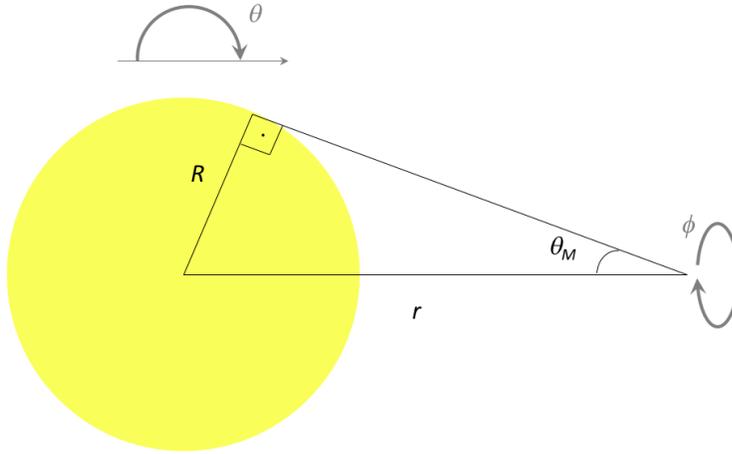


Figure 5.6: From a distance  $r$  away, a star of radius  $R$  is seen up to an angle  $\theta_M$ . The measured intensity is thus  $I$  for meridional angles  $0 \leq \theta \leq \theta_M$  and azimuthal angles  $0 \leq \phi \leq 2\pi$ , and zero for  $\theta_M < \theta \leq \pi$  and  $0 \leq \phi \leq 2\pi$ .

### 5.3.4 Solved problem

A spherical star of radius  $R$  emits radiation of intensity  $I_v$  in all directions (Fig. 5.6). For a distance  $r$ , describe the radiation field, obtain the mean intensity, and the flux.

1. Radiation Field: The observer sees an anisotropic radiation field, with the star subtending a finite angular size, and elsewhere having zero intensity. The field is thus

$$I_v(\theta, \phi) = I_v \quad \text{for } 0 \leq \theta \leq \theta_M \quad \text{and} \quad 0 \leq \phi \leq 2\pi \quad (5.44)$$

and zero otherwise; where  $\theta_M = \text{asin}(R/r)$ .

2. The mean intensity is

$$J_v = \frac{1}{4\pi} \oint I_v d\omega \quad (5.45)$$

$$= \frac{I_v}{4\pi} \int_0^{2\pi} d\phi \int_0^{\theta_M} \sin \theta d\theta \quad (5.46)$$

$$= \frac{1}{2} I_v (1 - \cos \theta_M) \quad (5.47)$$

$$= \frac{1}{2} \{1 - \cos [\text{asin}(R/r)]\} \quad (5.48)$$

using

$$1 - \cos [\text{asin}(R/r)] = \sqrt{1 - \sin^2 [\text{asin}(R/r)]} \quad (5.49)$$

$$= \sqrt{1 - (R/r)^2} \quad (5.50)$$

$$= \frac{1}{r} (r^2 - R^2)^{-1/2} \quad (5.51)$$

So

$$J_\nu = \frac{I_\nu}{2r} \left[ r - (r^2 - R^2)^{-1/2} \right] \quad (5.52)$$

3. The flux is

$$F_\nu = \int I_\nu \cos \theta d\omega = I_\nu \int_0^{2\pi} d\phi \int_0^{\theta_M} \cos \theta \sin \theta d\theta \quad (5.53)$$

Evaluating the integral

$$F_\nu = \pi I_\nu \sin^2 \theta_M = \pi I_\nu \left( \frac{R}{r} \right)^2 \quad (5.54)$$

So we recover the  $1/r^2$  dependency on distance.

## 5.4 Luminosities

We define the bolometric flux as the flux integrated in all wavelengths (Eq. 5.10).

$$F_{\text{bol}} = F = \int_0^\infty F_\nu d\nu \quad \text{and} \quad F = \frac{dE}{dAdt}$$

The quantity  $dE/dt$  in physics is called power, yet in astronomy we prefer to call it *luminosity*. So, Flux  $\times$  Area = Luminosity. For a source of constant luminosity, as most sources in astrophysics are (in the timescales we measure them), the product Flux  $\times$  Area is constant. Therefore, the flux falls with area following an inverse square law, recovering Eq. (5). For a spherical star, the area is  $4\pi R^2$  and we can write

$$L_\star = \text{Area} \times \text{Flux} = 4\pi R^2 F_\star. \quad (5.55)$$

where  $F_\star$  is the flux at the stellar surface.

### 5.4.1 Tying magnitudes and luminosities: bolometric correction

Notice that in Eq. (5.20) we wrote flux instead of luminosity, even though the idea of an absolute magnitude is to resolve the distance degeneracy so that we measure the stars' true brightness relative to one another.

The reason is that luminosity corresponds to the *bolometric* flux of the star, whereas magnitudes are usually measured within a specific range of wavelengths (a waveband).

Hipparchus and Ptolemy, working with the naked eye, did not have means to measure infrared or ultraviolet magnitudes. They measured it in the visible part of the electromagnetic spectrum only. Thus, these are also called *visible* magnitudes. We can call the apparent visible magnitude  $V$ , and the absolute visible magnitude  $M_V$ .

A *bolometric correction* is needed to convert these magnitudes to a bolometric magnitude that covers the whole spectrum and can be tied to luminosity.

$$M_{\text{bol}} = M_V + BC \quad (5.56)$$

For the Sun, the absolute visible magnitude is  $M_V = 4.83$ . The Sun's bolometric correction is  $BC = -0.09$ . The Sun has but a marginal bolometric correction because most of its energy is radiated in the visible. Stars much cooler or much hotter than the Sun can have significant bolometric corrections as most of their energy is radiated in other parts of the electromagnetic spectrum.

Having bolometric magnitudes, we can write

$$M_{\text{bol}} = -2.5 \log \left( \frac{L_\star}{4\pi D^2} \right) + C \quad (5.57)$$

And thus, in solar units, the stellar luminosity is

$$\frac{L}{L_\odot} = 10^{-0.4(M_{\text{bol}} - M_{\odot, \text{bol}})} \quad (5.58)$$

#### 5.4.2 The flux at the stellar surface: Stefan-Boltzmann law

Eq. (5.70) relates the luminosity with the stellar area and the stellar surface flux. While the radius is an intuitive quantity (although it is not always easily measurable), the flux at the stellar flux is not yet defined. To find this quantity, we use a concept of thermal and statistical physics, the Stefan-Boltzmann law of black body radiation.

Consider an ideal body that absorbs all incident radiation, i.e., a perfect absorber. If only the absorption is considered, then the body would appear black in any wavelength; therefore, it is called a *black body*. However, if the body is in thermal equilibrium, neither cooling down nor heating up, the same amount of energy absorbed must be re-emitted, i.e., in thermal equilibrium this body must also be a perfect emitter. The radiation emitted by a black body in thermal equilibrium is called *black body radiation*.

Because black bodies are idealized bodies, and only the temperature went into the above considerations, we can expect that black body radiation will depend only on temperature. The properties of black body radiation were found before quantum mechanics was understood. First empirically by Josef Stefan and then shown by Ludwig Boltzmann through thermodynamics, the energy density of black body radiation is

$$u = aT^4 \quad (5.59)$$

where the proportionality factor  $a$  is called *radiation constant*. Its numerical value is  $a = 7.5667 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$ .

Recalling how we found the mean intensity and energy density to be related (Eq. 5.40), we can write

$$\frac{4\pi}{c} J = aT^4 \quad (5.60)$$

For an isotropic field,  $I = J$  and the flux is  $F = \pi I$ , so we can write

$$F = \sigma T^4, \quad (5.61)$$

a result called Stefan-Boltzmann law. The constant  $\sigma = ac/4$  is called Stefan-Boltzmann constant. Its numerical value is

$$\sigma = 5.6704 \times 10^5 \text{ erg cm}^{-2} \text{ s}^{-1} \text{ K}^{-4}. \quad (5.62)$$

The same law was later derived by Max Planck in the work that opened the doors of quantum mechanics. The intensity of black body radiation was empirically known to be

$$B_\nu = \frac{C_1 \nu^3}{e^{C_2 \nu/T} - 1} \quad (5.63)$$

where  $C_1$  and  $C_2$  are constants. Equating the temperature of the gas with the temperature of the radiation field, and using the daring ansatz that the radiation field was composed of radiation “atoms” whose energy was proportional to their frequency, Planck found

$$B_\nu \equiv \frac{2\nu^2}{c^2} \frac{h\nu}{\exp(h\nu/kT) - 1} \quad (5.64)$$

where  $h$  is Planck’s constant, relating energy and frequency of the radiation ( $E = h\nu$ ), and  $k$  is Boltzmann constant, relating the energy and temperature of the gas ( $E = kT$ ).

Since the field is isotropic, the flux of the half-hemisphere is  $F_\nu = \pi B_\nu$ . To find the bolometric flux we integrate in frequency

$$F = \int_0^\infty B_\nu d\nu = \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3 d\nu}{\exp(h\nu/kT) - 1} \quad (5.65)$$

Substituting  $x = h\nu/kT$ , we have

$$F = \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \quad (5.66)$$

The integral can be evaluated to  $\pi^4/15$ . The result is thus the Stefan-Boltzmann law

$$F = \left(\frac{k^4 \pi^2}{60c^2 \hbar^3}\right) T^4 = \sigma T^4 \quad (5.67)$$

where the Stefan-Boltzmann constant in terms of more fundamental constants is

$$\sigma \equiv \left(\frac{k^4 \pi^2}{60c^2 \hbar^3}\right) \quad (5.68)$$

So, what does it all have to do with stars? Well, as it turns out, stars are really well approximated as black bodies.

To understand why, let us consider what goes into the definition of a black body. It is a perfect absorber and a perfect emitter, described only by temperature, irrespective of material. This means that if a blackbody is composed of matter and radiation, the radiation and the matter must be at the same temperature: the dynamical equilibrium of matter and the thermal equilibrium of radiation combine into a state of thermodynamical equilibrium. This is not as intuitive as it sounds. The air and radiation around us are never in thermodynamic equilibrium. At daylight, if you measure the temperature of

air, it will be around room temperature, 300K. Yet, the radiation is composed of photons that left the Sun, travelled through a near vacuum for 8 minutes until entering the Earth's atmosphere and scattering their way towards your retina. They are solar photons, and will have a spectrum of a body at the temperature of the solar photosphere, around 5800 K. Clearly, radiation and matter are at very different temperatures.

For black bodies, matter and radiation are at the same temperature: the photons have been thermalized. For matter, thermalization happens when collisions relax the molecules to maximum entropy. If collisions are what set this state of equilibrium, we can expect that a necessary condition is that the mean free path of the molecules is small compared to the length scales of the system. Likewise, if radiation and matter are supposed to have the same temperature, the condition should be that the mean free path of the photons should also be small. Earth's atmosphere does not satisfy this condition: it is *transparent* to solar photons, which prevents their thermalization. When the mean free path of the photons is small, absorptions and re-emissions thermalize them, and the radiation field becomes Planckian.

We will examine the physics of this process in more detail in the next sections, also showing that stars are in such conditions. The photons produced in the core have small mean free paths and only in a narrow layer close to the surface, less than 500 km thick (less than 0.1% of the solar radius) they can escape. Thermodynamical equilibrium is a valid approximation, and we can then write the stellar flux following Stefan-Boltzmann's law

$$F_{\star} = \sigma T_{\star}^4, \quad (5.69)$$

allowing us to finally write the luminosity as

$$L_{\star} = 4\pi R_{\star}^2 \sigma T_{\star}^4. \quad (5.70)$$

## 5.5 Wien's displacement law

Wien's displacement law states that the peak wavelength of a blackbody radiation, multiplied by its temperature, is constant.

$$\lambda_{\max} T \equiv \text{const} = 2.898 \times 10^{-3} \text{m K} \quad (5.71)$$

or, conversely, the peak frequency divided by the temperature, is constant

$$\frac{\nu_{\max}}{T} \equiv \text{const} = 5.879 \times 10^{10} \text{Hz K}^{-1}. \quad (5.72)$$

This law follows immediately from Planck's law, by taking the derivative with respect to frequency

$$\frac{c^2}{2h} \frac{dI_{\nu}}{d\nu} = \frac{3\nu^2}{\exp(h\nu/kT) - 1} - \frac{\nu^2}{[\exp(h\nu/kT) - 1]^2} e^{h\nu/kT} \frac{h}{KT} \quad (5.73)$$

and equating it to zero, we find the frequency  $\nu_{\max}$  where the intensity peaks.

$$\frac{\nu_M^2}{\exp(h\nu_M/kT) - 1} \left( 3 - \frac{\nu_M \exp(h\nu_M/kT)}{[\exp(h\nu_M/kT) - 1]} \frac{h}{KT} \right) = 0 \quad (5.74)$$

Substituting  $x = h\nu_M/kT$ , it reduces to

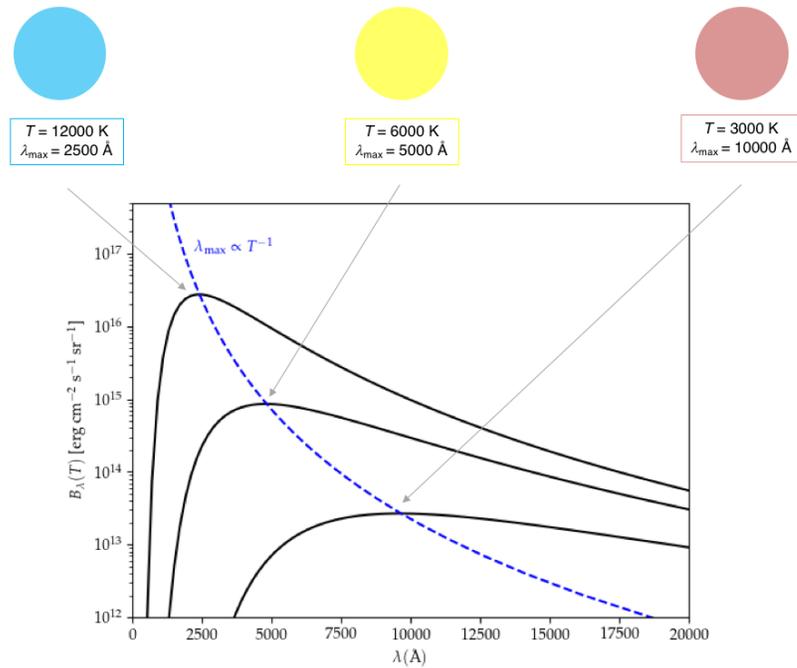


Figure 5.7: Wien's displacement law applied to stellar temperatures reveals that hot stars are blue and cooler stars are red.

$$xe^x - 3e^x + 3 = 0 \quad (5.75)$$

which we can solve numerically to find  $x \equiv \text{const}$ , i.e.,  $\nu_M/T \equiv \text{const}$ .

Eq. (5.70) and Eq. (5.71) already explain a lot of the observed properties of stars. The luminosity is a function of the radius (more emitting surface), so dwarfs are less luminous than giants. Betelgeuse is about 1000 times bigger than the Sun, so the area by itself would account for a million-fold factor in luminosity. The luminosity is also a very strong function of the temperature. A change in temperature by a factor 10, the range from the coldest ( $\approx 3000\text{K}$ ) to the hottest stars ( $\approx 30\,000\text{K}$ ), implies a  $10^4$  change in luminosity. Wien's displacement law implies that hot stars are blue, and cold ones are red (Fig. 5.7).

## 5.6 Kirchhoff three laws of spectroscopy

Gustav Robert Kirchhoff, not knowing about energy levels in the atom, coined the term "black body" radiation and also postulated three empirical laws that take his name. These laws are

1. A hot dense gas produces light with a continuous spectrum.
2. A hot tenuous gas produces light with emission lines at discrete wavelengths

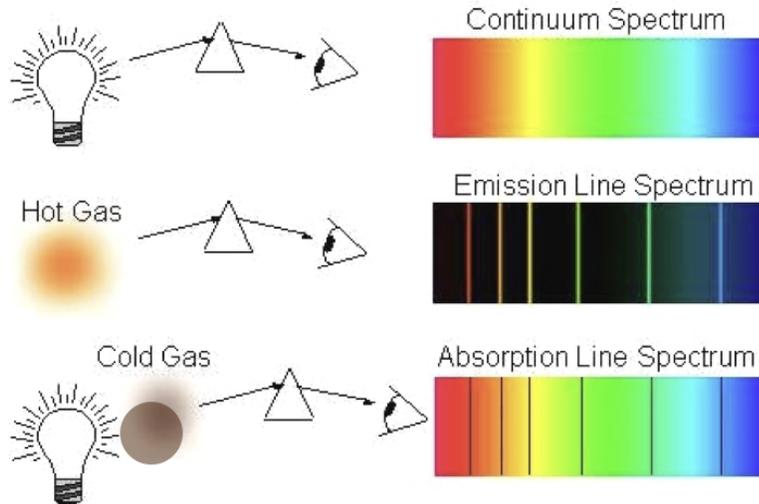


Figure 5.8: Kirchhoff's 3 empirical laws of spectroscopy. 1) A hot dense gas or solid emits radiation as a continuum. 2) A thin hot gas emits in discrete wavelengths. 3) A continuum seen through a colder gas will absorb in discrete wavelengths.

3. A hot dense gas surrounded by a cool tenuous gas produces light with a continuous spectrum which has gaps at discrete wavelengths.

An illustration of the laws is shown in Fig. 5.8. Let us understand them.

### 5.6.1 Emission and absorption of radiation

As a ray traverses a slab of material (Fig. 5.9), intensity will in general be both removed from the beam by absorption events and added to the beam by emission events. Let us examine these two processes isolatedly.

#### 5.6.1.1 Absorption, opacity, and optical depth

After traversing a purely absorbing medium of length  $ds$ , radiation of intensity  $I_\nu$  will come out on the other side as  $I_\nu + dI_\nu$ , where  $dI_\nu$  is negative. Intuitively, we can say that the amount of radiation removed will depend on the size of the slab, because a bigger slab means there is more material absorbing. Likewise, we can expect that it will depend on the density; a higher density means that there are more absorbers packed per volume. It should also depend on the intensity of the radiation; after all, if there are no incoming photons, there is nothing to be absorbed. Based on these notions, we can write

$$dI_\nu \propto -I_\nu \rho ds \quad (5.76)$$

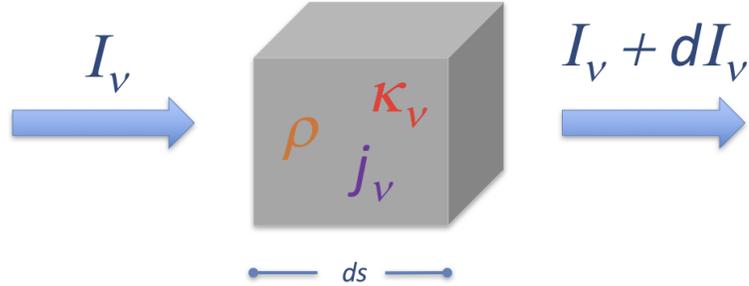


Figure 5.9: As radiation crosses a slab of matter of density  $\rho$ , intensity is either removed from the beam according to the opacity  $\kappa$ , or added according to the emissivity  $j$ .

we can replace the proportionality by an identity by defining the **opacity** as the proportionality factor, giving it the symbol  $\kappa_v$

$$dI_v = -I_v \kappa_v \rho ds. \quad (5.77)$$

The opacity quantifies the resistance of matter to the passage of radiation. It depends on frequency, because a material can be at the same time transparent in one wavelength while opaque in another. Glass, for instance, is transparent in visible wavelengths and opaque in infrared.

Frequently we also use

$$dI_v = -\alpha_v I_v ds \quad (5.78)$$

where  $\alpha_v = \kappa_v \rho$  is called absorption coefficient.

Notice that we can easily integrate Eq. (5.77) by dividing both sides by  $I_v$

$$\frac{dI_v}{I_v} = -\kappa_v \rho ds \quad (5.79)$$

which integrates to

$$I_v = I_v(0) \exp\left(-\int \kappa_v \rho ds\right) \quad (5.80)$$

We can also define the **optical depth** infinitesimal

$$d\tau_v = \kappa_v \rho ds \quad (5.81)$$

and thus the optical depth as the path integral of the opacity. So, the absorption is

$$I_v = I_v(0) e^{-\tau_v} \quad (5.82)$$

Radiation is attenuated as it travels through the medium. The attenuation increases rapidly with optical depth. For  $\tau = 5$  only 0.007 of the original radiation escapes. For  $\tau = 10$  is it  $5 \times 10^{-5}$ .

### 5.6.1.2 Origin of opacity

The absorption law is phenomenological. To give it an intuitive physical grounding, imagine an ensemble of absorbers of number density  $n$ , all having cross section  $\sigma_\nu$ . The total number of absorbers in a volume  $dV$  is

$$N = ndV \quad (5.83)$$

and collectively they offer an absorbing area  $dA = ndV\sigma_\nu$ . So, the amount of energy absorbed from the beam is

$$dE_\nu = -dI_\nu dA d\omega dt dv \quad (5.84)$$

$$\propto (ndV\sigma_\nu) d\omega dt dv \quad (5.85)$$

The proportionality constant has to have dimension of intensity, and whatever constant present can be absorbed into  $\sigma_\nu$ . Thus

$$dE_\nu = I_\nu (ndV\sigma_\nu) d\omega dt dv \quad (5.86)$$

writing  $dV = dA ds$ , and comparing Eq. (5.84) and Eq. (5.86),

$$dI_\nu = -n\sigma_\nu I_\nu ds \quad (5.87)$$

So

$$\kappa_\nu \rho = n\sigma_\nu \quad (5.88)$$

i.e., the opacity is a measurement of the density of absorbing material.

### 5.6.2 Optical depth as mean free path

Let us show that  $\tau_\nu = 1$  means one photon mean free path. I.e., we want to show that the following is true

$$\tau_\nu = 1 = \int_0^{l_{\text{mfp}}} \kappa_\nu \rho ds, \quad (5.89)$$

where  $l_{\text{mfp}}$  is one photon mean free path. Consider the average optical depth over which a photon travels before being absorbed

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu p(\tau_\nu) d\tau_\nu, \quad (5.90)$$

where  $p(\tau_\nu) d\tau_\nu$  is the probability of being absorbed in the interval  $(\tau_\nu, \tau_\nu + d\tau_\nu)$ , after having traveled  $(0, \tau_\nu)$ , before being absorbed. In other words, it is the probability of not being absorbed in  $(0, \tau_\nu)$  and being absorbed in  $d\tau_\nu$ .

Let us construct a mathematical form for the probability of absorption. It must be zero if there is no absorption, and one if there is. The following function is appropriate

$$p = \frac{\Delta I(\tau_\nu)}{I_0} \quad (5.91)$$

as it is 0 for  $\Delta I = 0$  (not absorbed) and 1 for  $\Delta I = I$  (absorbed). Thus

$$p = \frac{\Delta I(\tau_v)}{I_0} = \frac{I_0 - I(\tau_v)}{I_0} = 1 - \frac{I(\tau_v)}{I_0} \quad (5.92)$$

The probability  $p_1$  that the photon is not absorbed until  $\tau_v$  is

$$p_1 = 1 - p = \frac{I(\tau_v)}{I_0} = e^{-\tau_v}. \quad (5.93)$$

The probability  $p_2$  that the photon is absorbed in  $(\tau_v, \tau_v + d\tau_v)$  is

$$p_2 = \frac{\Delta I(\tau_v, \tau_v + d\tau_v)}{I(\tau_v)} \quad (5.94)$$

$$= \frac{dI_v}{I(\tau_v)} = d\tau_v \quad (5.95)$$

So, the total probability is  $p_1 \times p_2$

$$\text{total probability : (Not absorbed in } [0, \tau_v]) \times (\text{absorbed in } [\tau_v, \tau_v + d\tau_v]) = e^{-\tau_v} d\tau_v \quad (5.96)$$

Therefore,

$$\langle \tau_v \rangle = \int_0^{\infty} \tau_v p(\tau_v) d\tau_v = \int_0^{\infty} \tau_v e^{-\tau} d\tau_v \quad (5.97)$$

$$= -(1+x)e^{-x}|_0^{\infty} = 0 + (1)e^0 = 1 \quad (5.98)$$

So, we arrive at the condition we wanted to prove. The average optical depth traveled before being absorbed is  $\langle \tau_v \rangle = 1$ . This corresponds to the mean free path of a photon.

Notice that, for a homogeneous material,  $\kappa\rho \equiv \text{const}$

$$\langle \tau_v \rangle = 1 = \int \kappa\rho dz = \kappa\rho l_{\text{mfp}} \quad (5.99)$$

So we arrive at an enlightening relation

$$l_{\text{mfp}} = \frac{1}{\kappa\rho} \quad (5.100)$$

the opacity  $\alpha = \kappa\rho$  is the inverse of the mean free path.

### 5.6.3 Emission

So far we talked about absorption only. The opposite process, emission, should contribute energy to the outgoing beam. Looking again at Fig. 5.9, we can parametrize emission by being proportional to the density and to the length of the material,

$$dI_v \propto \rho ds \quad (5.101)$$

with  $dI_v > 0$ . Again, to make it an equality, we define a quantity,  $j_v$ , that we will call the *emissivity*

$$dI_v = j_v \rho ds. \quad (5.102)$$

In terms of energy, if we consider  $dI_\nu = dE_\nu/(dA\nu d\omega dt)$ , we

$$dE_\nu = j_\nu \rho dV d\nu d\omega dt \quad (5.103)$$

where we substituted  $dV = dA ds$ . The unit of emissivity is thus

$$[j_\nu] = \text{erg s}^{-1} \text{g}^{-1} \text{ster}^{-1} \text{Hz}^{-1} \quad (5.104)$$

we see that  $j_\nu$  means power per mass. It measures how much energy per time a gram of material contributes in a particular wavelength and direction. Notice that when multiplied by density this becomes power per volume

$$[j_\nu \rho] = \text{erg s}^{-1} \text{cm}^{-3} \text{ster}^{-1} \text{Hz}^{-1}, \quad (5.105)$$

now measuring how much energy per time every cubic centimeter of material contributes in a particular wavelength and direction. It is useful to define this quantity, the emissivity per volume, as

$$\chi_\nu = j_\nu \rho. \quad (5.106)$$

#### 5.6.4 The radiative transfer equation

We found how absorption and emission add intensity to a beam. Let us now combine them both

$$dI_\nu = -\kappa_\nu I_\nu ds + j_\nu \rho ds \quad (5.107)$$

dividing both sides by  $\kappa_\nu \rho ds$ ,

$$\frac{dI_\nu}{\kappa_\nu \rho ds} = -I_\nu + \frac{j_\nu}{\kappa_\nu} \quad (5.108)$$

substituting (Eq. 5.81) for the optical depth, and defining  $S_\nu \equiv j_\nu/\kappa_\nu$ , we arrive at

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu} \quad (5.109)$$

This is the fundamental equation of radiation transfer. The function  $S_\nu$  is called the **source function**.

Notice that in the special case of black body radiation  $I_\nu = B_\nu$ , and because the body is in equilibrium,  $dI_\nu = 0$ . So, the source function for black bodies is  $S_\nu = B_\nu$ .

The general solution of the equation is found by using the integration factor  $e^{\tau_\nu}$  on both sides

$$e^{\tau_\nu} dI_\nu + e^{\tau_\nu} I_\nu d\tau_\nu = S_\nu e^{\tau_\nu} d\tau_\nu \quad (5.110)$$

$$d(e^{\tau_\nu} I_\nu) = S_\nu e^{\tau_\nu} d\tau_\nu \quad (5.111)$$

Integrating between 0 and  $\tau_\nu$

$$e^{\tau_\nu} I_\nu(\tau_\nu) - I_\nu(0) = \int_0^{\tau_\nu} S_\nu(t_\nu) e^{t_\nu} dt_\nu \quad (5.112)$$

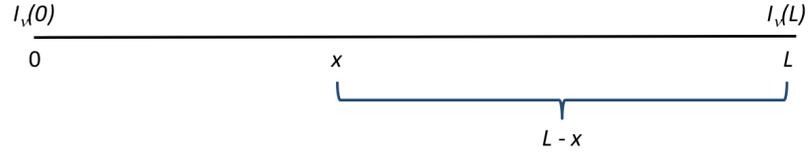


Figure 5.10: The intensity  $I_v(L)$  in a beam of original intensity  $I_v(0)$  after traversing a path of length  $L$  is  $I_v(L) = I_v(0)e^{-\kappa_v \rho L} + \int_0^L j_v \rho e^{-\kappa_v \rho(L-x)} dx$ . The original intensity is attenuated, and intensity is added by emission. A point  $x$  along the path contributes with emission that will also suffer absorption along the remainder  $(L-x)$  of the path that it has to travel. The total emission is found by integrating along the whole path.

Finally, dividing by  $e^{\tau_v}$  we arrive at the *general solution* for the equation of radiative transfer

$$I_v(\tau_v) = I_v(0)e^{-\tau_v} + \int_0^{\tau_v} S_v(t_v)e^{-(\tau_v-t_v)} dt_v \quad (5.113)$$

The 1st term is the extinction of the original intensity, and the 2nd is the emission at a point  $t_v$ , extinguished in the path from  $t_v$  to  $\tau_v$ .

The meaning of the radiative transfer becomes simpler to understand when we pass back from optical depth to geometrical depth. Writing  $\tau_v = \kappa_v \rho L$  and  $t_v = \kappa_v \rho x$ ,

$$I_v(L) = I_v(0)e^{-\kappa_v \rho L} + \int_0^L j_v \rho e^{-\kappa_v \rho(L-x)} dx \quad (5.114)$$

As radiation traverses a distance  $L$  through a medium of opacity  $\kappa$  and density  $\rho$ , the original intensity is attenuated by  $\exp(-\kappa \rho L)$ . Along the way, we also have to take into account emission, that adds energy to the beam. Emission from every point  $x$  will reach  $L$  attenuated by absorption through the length  $L-x$  it had to traverse.

### 5.6.5 Limits

Let us consider some limits that will illustrate some cases of physical interest. Consider first the *optically thin* limit, of  $\tau_v \ll 1$ . In this case we can approximate

$$e^{-\tau_v} \rightarrow 1 - \tau_v \quad (5.115)$$

applying it to Eq. (5.113) and assuming  $j_v \rho$  to be constant

$$I_v(L) = I_v(0) + j_v \rho L \quad (5.116)$$

The result shows that in the optically thin case, the original intensity is unaltered, but emission was added. The emission is unattenuated, so all contributions from 0 to the full geometrical depth  $L$  are added. In terms of the optical depth, this becomes

$$I_v(\tau_v) = I_v(0) + S_v \tau_v \quad (5.117)$$

In the *optically thick* limit,  $\tau_v \gg 1$ , the equation of radiative transfer becomes

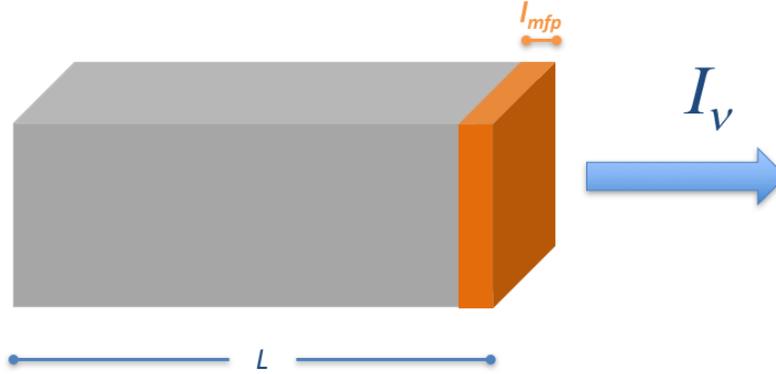


Figure 5.11: In the optically thick limit, the radiation is heavily absorbed and only material within one photon mean free path contributes to the emission.

$$I_\nu = S_\nu = \frac{j_\nu}{\kappa_\nu} = \frac{j_\nu \kappa_\nu \rho}{\kappa_\nu \kappa_\nu \rho} = j_\nu \rho \left( \frac{1}{\kappa_\nu \rho} \right) \quad (5.118)$$

$$I_\nu = j_\nu \bar{l} \quad (5.119)$$

Meaning that only stuff at 1 mean free path contributes to the emission (Fig. 5.11).

### 5.6.6 Kirchhoff's laws and the equation of radiative transfer

We are now in position to understand Kirchhoff's laws in terms of the equation of radiative transfer.

Consider the case  $I_\nu(0) = 0$ . There is no back illumination, only hot gas emitting. In this case, the equation becomes

$$I_\nu = S_\nu (1 - e^{-\tau_\nu}) \quad (5.120)$$

The 1st law is derived from this in the optically thick case ( $\tau_\nu \gg 1$ ), which leads to

$$I_\nu = S_\nu \quad (1\text{st law}) \quad (5.121)$$

We will see later that in the optically thick case the source function approaches a black body, so  $S_\nu$  is the Planck function, and  $I_\nu = B_\nu$ .

The 2nd law is also derived from this, but in the optically thin case ( $e^{-\tau_\nu} \approx 1 - \tau_\nu$ ), so

$$I_\nu = S_\nu \tau_\nu \quad (2\text{nd law}) \quad (5.122)$$

The intensity will be high where  $\tau_\nu$  is high. Since there is no background intensity, these are seen as emission lines.

The 3rd law is the case where there is a background intensity but no emission. The background source is a hot gas of intensity  $B_\nu$ , whereas the cold gas has  $S_\nu = 0$ . Thus

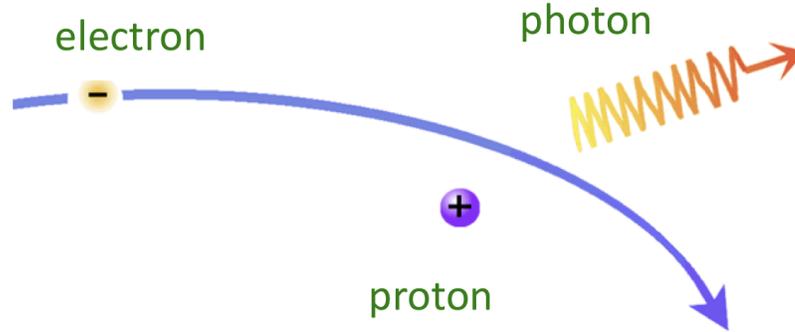


Figure 5.12: An electron accelerated in the electric field of a proton will emit radiation according to Larmor formula. This radiation is known as *Bremsstrahlung* (braking radiation).

$$I_\nu = B_\nu e^{-\tau_\nu} \quad (3\text{rd law}) \quad (5.123)$$

The intensity is a Planck continuum, lowered where  $\tau_\nu$  is high. Therefore, we see absorption lines.

## 5.7 Understanding the formation of the continuum

### 5.7.1 Bremsstrahlung

Bremsstrahlung is German for “braking radiation”. As derived from electromagnetism, a charge under acceleration will radiate. So, let us consider a simple system: a proton and an electron. Consider the reference frame of the proton. As the electron passes, it will be deflected by the proton’s electric field, as sketched in Fig. 5.12.

Consider that the electron is traveling with velocity  $u$ , and with impact parameter  $b$  with respect to the proton. The interaction time is thus  $t = b/u$ . The acceleration is constant, given by Coulomb law

$$a \approx \frac{e^2}{m_e b^2} \quad (5.124)$$

Larmor’s formula states that the radiated power is

$$P = \frac{2e^2 a^2}{3c^3} \approx \frac{e^6}{m_e^2 c^3 b^4} \quad (5.125)$$

The characteristic time leads to a characteristic frequency

$$\nu \sim \frac{1}{t} = \frac{u}{b} \quad (5.126)$$

As a crude approximation, the power spectrum is the power per frequency, thus

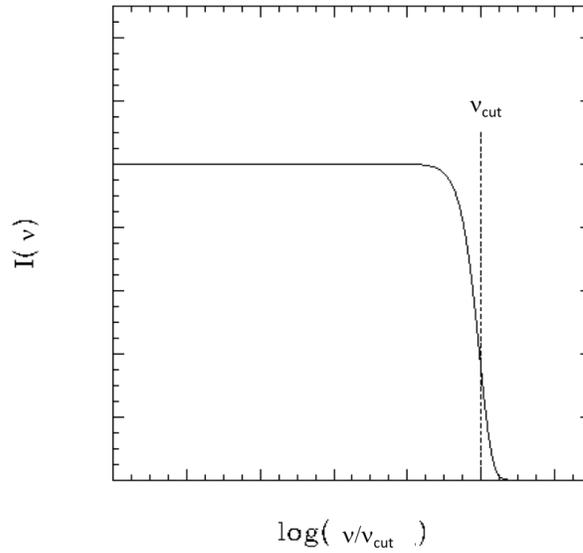


Figure 5.13: The spectrum of Bremsstrahlung. The emission is flat, up to a cutoff frequency. The cutoff frequency occurs because an electron cannot emit a photon of higher energy than its kinetic energy.

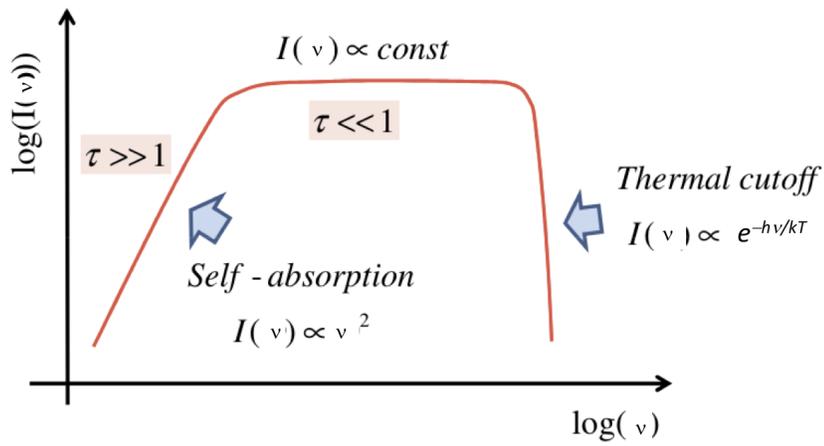


Figure 5.14: When Bremsstrahlung is emitted by an ensemble of particles, self-absorption has to be taken into account. This leads to the three conspicuous features of Bremsstrahlung: the self-absorption tail, the plateau of optically thin emission, and the thermal cutoff.

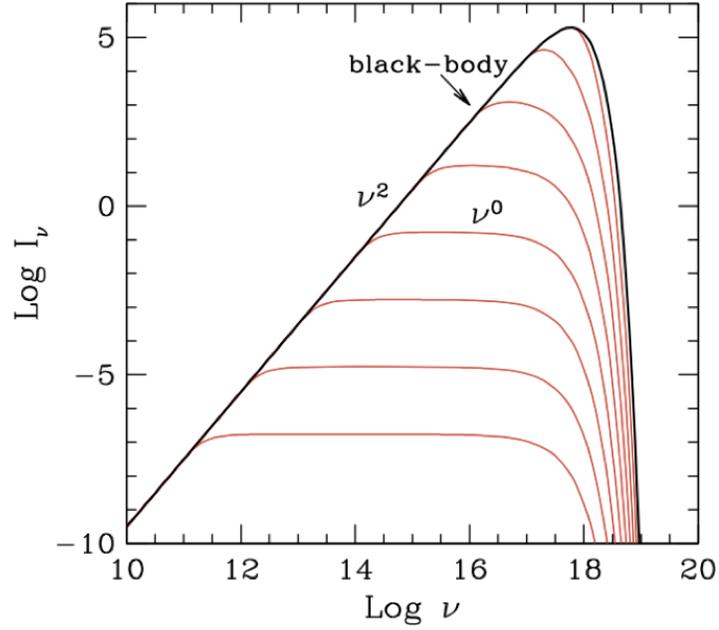


Figure 5.15: The formation of the blackbody continuum from Bremsstrahlung. As one increases the density, the optically thin plateau progressively disappears. When the gas is optically thick, the whole spectrum is self-absorbed and the spectrum becomes a blackbody.

$$P(\nu) \sim \frac{P}{\nu} = \frac{e^6}{m_e^2 c^3 u b^3} \quad (5.127)$$

To pass from an isolated system to an ensemble of particles, consider that the impact parameter is given by the average separation of the protons. Given the number density of protons  $n_p$ , in units of  $\text{cm}^{-3}$ , the length we can construct is  $l = n_p^{-1/3}$ . This will give the impact parameter

$$b \approx n_p^{-1/3} \quad (5.128)$$

So,

$$P(\nu) \approx \frac{e^6 n_p}{m_e^2 c^3 u} \quad (5.129)$$

We can state the power (luminosity) as a function of energy,  $P = dE/dt$ . The power per frequency is thus

$$P(\nu) = \frac{dE}{dt d\nu} \approx \frac{e^6 n_p}{m_e^2 c^3 u} \quad (5.130)$$

This is the power for one single electron. To convert to emissivity, we recall the definition of that quantity (Eq. 5.103)

$$j_\nu \rho = \frac{dE}{dV d\nu dt d\omega} \quad (5.131)$$

So, we have from Eq. (5.130) the spectral power  $dE/d\nu dt$ . Comparing with Eq. (5.131) we need to reach something with dimension of  $dE/d\nu dt dV d\omega$ . The factors of volume  $dV$  and of solid angle  $d\omega$  are missing. For  $dV$  we multiply by the volume density of emitters (the electron density). For  $d\omega$  we assume isotropy and divide by  $4\pi$ . So,

$$j_\nu \rho = \frac{n_e}{4\pi} \frac{dE}{d\nu dt} \quad (5.132)$$

Substituting Eq. (5.130), we find an expression for the ensemble emissivity.

$$j_\nu \rho = \frac{n_e n_p}{4\pi} \frac{e^6}{m_e^2 c^3} \frac{1}{u}. \quad (5.133)$$

### 5.7.2 Thermal Bremsstrahlung

Notice that Eq. (5.133) still depends on the velocity  $u$ . In an ensemble, particles will have different velocities, so Eq. (5.133) cannot be the whole story. To get the emissivity of the ensemble, we need to integrate over the velocity distribution. In thermodynamical equilibrium, the number of particles with velocity between  $u$  and  $du$  is given by the Maxwell-Boltzmann distribution

$$N_e(u) du = 4\pi N_e \left( \frac{m_e}{2\pi kT} \right)^{3/2} u^2 \exp\left(-\frac{m_e u^2}{2kT}\right) du \quad (5.134)$$

according to Eq. (5.133) an electron with  $u \rightarrow 0$  should contribute an infinite emissivity (because its interaction time is so long). Yet, the number of such electrons according to Eq. (5.134) is also very low, so the emission will be finite. This shows that to find the total emissivity, we need to weight Eq. (5.133) by Eq. (5.134). Writing  $\chi = j\rho$ , we have

$$\chi_\nu(T) \equiv \frac{\int \chi_\nu(u) N_e(u) du}{\int N_e(u) du} \quad (5.135)$$

substituting Eq. (5.133) and Eq. (5.134)

$$\chi_\nu(T) = \frac{n_e n_p}{4\pi} \frac{e^6}{m_e^2 c^3} \frac{\int_0^\infty u \exp\left(-\frac{m_e u^2}{2kT}\right) du}{\int_0^\infty u^2 \exp\left(-\frac{m_e u^2}{2kT}\right) du} \quad (5.136)$$

The integral yields

$$\chi_\nu(T) = \left( \frac{1}{2\pi m_e} \right)^{3/2} \frac{e^6}{k^{1/2} c^3} n_e n_p T^{-1/2} \quad (5.137)$$

This is an interesting result. The emissivity does not depend on frequency. It is completely flat. As electrons are accelerated in the electric field of protons, they emit photons of *all* frequencies, at the *same* rate.

Yet, in light of quantum physics, such result cannot be the correct. The photon energy is proportional to the frequency. An electron cannot radiate a photon of energy higher than its kinetic energy. This will lead to an exponential cutoff in the emission.

$$\chi_\nu(T) = \left( \frac{1}{2\pi m_e} \right)^{3/2} \frac{e^6}{k^{1/2} c^3} n_e n_p T^{-1/2} e^{-h\nu/kT} \quad (5.138)$$

The spectrum is shown in Fig. 5.13. A full quantum calculation will also contribute some other constants, but the form is the same. Considering a generic ion of charge  $Ze$  and number density  $n_i$  contributing  $Z$  electrons, the emissivity is

$$\chi_\nu \equiv \frac{2^5 \pi e^6}{3mc^3} \left( \frac{2\pi}{3km} \right)^{1/2} T^{-1/2} Z^2 n_e n_i \exp\left(-\frac{h\nu}{kT}\right) \quad (5.139)$$

### 5.7.3 From Bremsstrahlung to Blackbody

So far we talked about emission. Now let us consider absorption. Due to Kirchhoff's 1st law

$$\chi_\nu = \alpha_\nu B_\nu \quad (5.140)$$

And, according to the previous result

$$\frac{dE}{dt dV d\nu} = 4\pi \chi_\nu \quad (5.141)$$

So we can write

$$\alpha_\nu = \frac{\chi_\nu}{B_\nu} = \frac{4e^6}{3mhc} \left( \frac{2\pi}{3km} \right)^{1/2} T^{-1/2} Z^2 n_e n_i \nu^{-3} (1 - e^{-h\nu/kT}) \quad (5.142)$$

For the Rayleigh-Jeans tail of low energy

$$\alpha_\nu = \frac{4e^6}{3kc} \left( \frac{2\pi}{3km} \right)^{1/2} T^{-3/2} Z^2 n_e n_i \nu^{-2} \quad (5.143)$$

So,  $B_\nu = \chi_\nu/\alpha_\nu$ , will have a  $\nu^2$  tail in low frequencies because of absorption, and an exponential tail because of emission.

The conspicuous features in the Bremsstrahlung spectrum (Fig. 5.14) are (1) the  $\nu^2$  self-absorption: Bremsstrahlung emits but self-absorbs as the medium gets thicker (2) the flat plateau of frequency-independent optically thin emission and (3) the thermal exponential cutoff in high-frequencies. The cutoff happens because an electron cannot emit a photon of energy higher than its kinetic energy.

As the medium gets optically thicker, the self-absorption starts to "eat away" the  $\nu_0$  flat part of the spectrum, and Bremsstrahlung becomes blackbody. The self-absorption corresponds to optical depths  $\tau \gg 1$ . The  $\nu^{-3}$  term ensures that it happens mostly at low frequencies. Increasing the density the spectrum is self-absorbed to higher and higher frequencies. When *all* the spectrum is self-absorbed ( $\tau_\nu > 1$  for all  $\nu$ ) and the distribution is Maxwellian, then we have a blackbody curve. Note that Bremsstrahlung depends on density, but blackbody does not. This is because we receive radiation from one optical depth only. The absorption decreases as the density increases, but so does the emissivity.

## 5.8 Microphysics of Radiative Transfer

### 5.8.1 Boltzmann ratios

In local thermodynamical equilibrium (LTE), the ratio  $n_1$  and  $n_2$  between the occupancy of excitation levels is given by Boltzmann's law

$$P(E) = g(E)e^{-E/kT} \quad (5.144)$$

where  $P(E)$  is the probability of the system being at the state of energy  $E$ , and  $g(E)$  is the degeneracy of that level, its statistical weight.

Eq. (5.144) can be derived by the definition of entropy

$$S = -k \sum_i p_i \ln p_i \quad (5.145)$$

with the constrains that  $\sum_i p_i = 1$  and  $\langle E \rangle = \sum_i p_i E_i = U$  where  $U$  is the total energy. The probability  $p_i$  is the probability that maximizes the entropy. We rewrite the equation above using Lagrange multipliers

$$S = -k \sum_i p_i \ln p_i + \lambda_1 \left( \sum_i p_i - 1 \right) + \lambda_2 \left( \sum_i p_i E_i - U \right) \quad (5.146)$$

We find  $\lambda_2$  directly by derivating with respect to  $U$ , and applying the second law of thermodynamics

$$\frac{dS}{dU} = -\lambda_2 = \frac{1}{T} \quad (5.147)$$

Whereas derivating with respect to  $p_i$ , which should be zero (probability that maximizes entropy) yields

$$\frac{dS}{dp_i} = 0 = -k \ln p_i - k + \lambda_1 - \frac{E_i}{T} \quad (5.148)$$

Now isolate  $p_i$

$$p_i = \exp \left[ \frac{1}{k} \left( -1 + \lambda_1 - \frac{E_i}{T} \right) \right] \quad (5.149)$$

$$= \exp \left[ \frac{1}{k} (\lambda_1 - 1) \right] e^{-E_i/kT} \quad (5.150)$$

To find  $\lambda_1$  we now apply  $\sum_i p_i = 1$

$$\exp \left[ \frac{1}{k} (\lambda_1 - 1) \right] \sum_i e^{-E_i/kT} = 1 \quad (5.151)$$

that is

$$\lambda_1 = 1 - k \ln Z \quad (5.152)$$

where

$$Z = \sum_i e^{-E_i/kT} \quad (5.153)$$

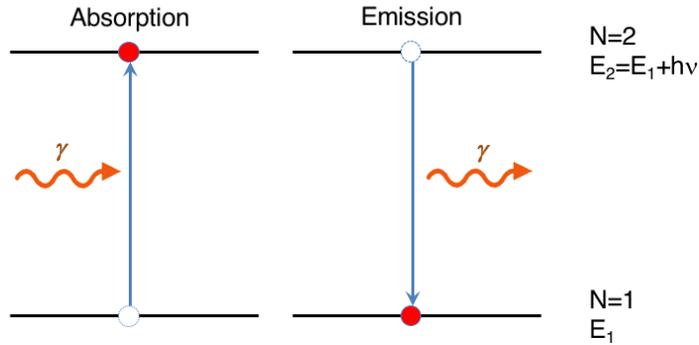


Figure 5.16: In a simple system of two levels, the possible events are absorption and emission.

is the partition function. We can also write

$$p_i = \frac{1}{Z} e^{-E_i/KT} \quad (5.154)$$

### 5.8.2 Relationship between absorption and emission

Radiative equilibrium means that all processes are in detailed balance (emission and absorption, excitation and de-excitation, ionization and recombination). No change in the quantities occurs. Considering Kirchhoff's 1st law,  $S_\nu = B_\nu$ , but recalling the definition of the source function,  $S_\nu = j_\nu/\kappa_\nu$ , the law reveals a connection between emission and absorption

$$j_\nu = \kappa_\nu B_\nu \quad (5.155)$$

This form of Kirchhoff's 1st law implies some relationship between emission and absorption at the microscopic level.

This relationship was unveiled by Einstein, via a very elegant consideration. Consider two discrete energy levels, 1 the lower energy and 2 the higher energy. The processes that can happen, as shown in Fig. 5.16, are

- Emission (2-1) : The system passes from 2 to 1 emitting a photon.
- Absorption (1-2) : The systems passes from 1 to 2 via radiative excitation, absorbing a photon.

In equilibrium, we must have the same number of emissions and absorptions. The number of emissions is given by how many atoms are in the second level,  $n_2$ , multiplied by the probability of emission, that we call  $A$ . The number of radiative excitations

depend on the number of atoms in the first level,  $n_1$ , the number of photons, which is given by the intensity  $I$ , and the probability of absorption, that we call  $B$ .

$$n_1 B I = n_2 A \quad (5.156)$$

Isolating  $I$ ,

$$I = \frac{n_2 A}{n_1 B} \quad (5.157)$$

In LTE, the population of a level  $n_1$  is given by the Boltzmann relation

$$n_i \propto g_i e^{-E_i/kT} \quad (5.158)$$

where  $g_i$  is the statistical weight of the level, and  $E_i$  its energy. Given that the states 1 and 2 are separated by the energy of a photon,  $h\nu$ , the ratio  $n_1/n_2$  is

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E_1/kT}}{g_2 e^{-(E_1+h\nu)/kT}} = \frac{g_1}{g_2} e^{h\nu/kT} \quad (5.159)$$

So,

$$I_\nu = \frac{A g_2}{B g_1} \frac{1}{e^{h\nu/kT}} \quad (5.160)$$

For a black body, the intensity must equal the Planckian, i.e.,

$$I_\nu = B_\nu. \quad (5.161)$$

And the Planckian should be

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad (5.162)$$

Comparing Eq. (5.160) and Eq. (5.162) shows that the mathematical form of the two equations is different. There is a minus one missing in the denominator in Eq. (5.160), so there is no way that the model can be complete. The missing minus one means that there is a term missing in Eq. (5.156).

Einstein reasoned that the missing term means that there is some process that we are ignoring. The way we isolated  $I$  in Eq. (5.157) means that if that equation has a  $-1$  in the denominator, there must be a negative term in the left hand side of Eq. (5.156), also proportional to  $I$ , i.e.,

$$(n_1 B - C) I = n_2 A \quad (5.163)$$

where  $C$  is some multiple of the probability of occurrence of this process. The fact that the probability is negative shows that physically, this term is in the wrong side of the equation. It should actually be in the right hand side

$$n_1 B I = n_2 A + C I \quad (5.164)$$

Isolating  $I$

$$I = \frac{n_2 A}{n_1 B - C} \quad (5.165)$$

let us find the exponential. Given Boltzmann formula, the exponential is proportional to the  $n_2/n_1$  factor. If we divide both the numerator and denominator by  $n_2$

$$I = \frac{A}{n_1/n_2 B - C/n_2} \quad (5.166)$$

And substituting Boltzmann formula,

$$I = \frac{A}{g_1/g_2 B e^{h\nu/kT} - C/n_2} \quad (5.167)$$

To make a term of  $-1$  appear in the denominator, we divide and multiply the denominator by  $C/n_2$

$$I = \frac{An_2/C}{n_2/C g_1/g_2 B e^{h\nu/kT} - 1} \quad (5.168)$$

We see that if we could remove the factors of  $n_2$  following  $C$ , Eq. (5.167) would have the functional form of the Planckian. So let us substitute  $C = Dn_2$ . Thus

$$I = \frac{A/D}{B/D g_1/g_2 e^{h\nu/kT} - 1} \quad (5.169)$$

and this now has the form of the Planckian if

$$\frac{A}{D} = \frac{2h\nu^3}{c^2} \quad (5.170)$$

$$\frac{B}{D} = \frac{g_2}{g_1} \quad (5.171)$$

### 5.8.2.1 Stimulated Emission

Let us understand what setting  $C = Dn_2$  means, back in Eq. (5.164).

$$n_1 B I = n_2 A + n_2 D I \quad (5.172)$$

The process that is missing is proportional to the intensity, so it is an absorption process. Yet, it is proportional also to the number of atoms in the level 2. Since we assumed only 2 levels, and level 2 is the most energetic, the process cannot be absorption: there is no upper level to jump to. The process is some process that involves jumping from 2 to 1, an emission process, yet somehow dependent on the radiation field. This term seems to tell us that the presence of a photon in the radiation field of exact the same frequency as the transition from 2 to 1 will somehow lead to emission of photon. Einstein called it *stimulated emission* (Fig. 5.17). A photon in the radiation field, if captured by an electron in the  $N = 1$  level, will lead to radiative absorption, happening with probability B. Yet, for an electron in the  $N = 2$  level, the same photon may lead to stimulated emission, happening with probability D.

To keep the number of symbols to a minimum, we keep the symbol B for transitions dependent on the radiative field, but give them subscripts indicating the jump that occurred. So, we write  $B \rightarrow B_{21}$ , meaning radiative transition from 1 to 2 (absorption), and  $D \rightarrow B_{21}$ , meaning radiative transition from 2 to 1 (stimulated emission). To keep symmetry, we also give subscripts to A, the emission probability, now called *spontaneous emission*, since it does not depend on the radiative field,  $A \rightarrow A_{21}$ . These coefficients are known as *Einstein coefficients*.

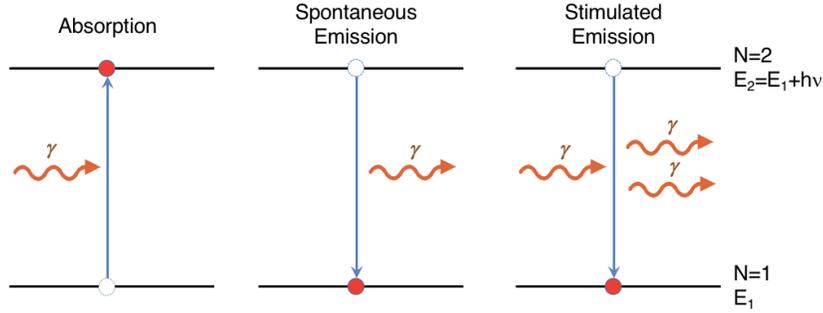


Figure 5.17: In radiative equilibrium, the simple 2-level system can only reproduce the Planckian if a 3rd process is considered, stimulated emission, in which a photon in the radiation field induces an electron in the  $N = 2$  level to emit an identical photon.

$$n_1 B_{12} I = n_2 A_{21} + n_2 B_{21} I \quad (5.173)$$

And now

$$\frac{A_{21}}{B_{21}} = \frac{2h\nu^3}{c^2} \quad (5.174)$$

$$\frac{B_{12}}{B_{21}} = \frac{g_2}{g_1} \quad (5.175)$$

these two equations above are called the Einstein relations. If these relations are obeyed, then the radiation field in this system of two levels reproduces the Planckian for all temperatures.

Notice that the fact that mathematically one can simply put the stimulated emission in the left-hand side of Eq. (5.163) means that the process can also be thought of as *negative absorption*. Due to interaction between the radiation field and matter, a photon was *added* to the field, with the same frequency that would normally be absorbed. The fact that intensity also encoded direction means that the photon that is added is *in the same direction* as the incoming photon. It intensifies the radiation field.

Modern technology uses this process every day, in the form of light amplification by stimulated emission of radiation, or *laser*. The fact that such a useful process came from the few lines of analysis above is mind-boggling.

### 5.8.3 The microphysics of the source function

Let us consider the energy absorbed. The absorption law states that the energy removed from the beam is  $\kappa_\nu \rho I_\nu$ . Microphysically, the energy removed is the number of absorption events, times the energy of each photon

$$\kappa_\nu \rho I_\nu = (n_1 B_{12} I_\nu) \times (h\nu) \quad (5.176)$$

Similarly, the energy emitted is the number of emissions, times the energy of each photon

$$j_\nu \rho = h\nu \times (n_2 B_{21} I_\nu + n_2 A_{21}) \quad (5.177)$$

We can treat stimulated emission as negative absorption and write the *effective* absorption

$$\kappa_\nu \rho I_\nu = h\nu I_\nu (n_1 B_{12} - n_2 B_{21}) \quad (5.178)$$

and thus the effective emission as

$$j_\nu \rho I_\nu = h\nu n_2 A_{21} I_\nu \quad (5.179)$$

Crossing  $I_\nu$  and using the Einstein relation  $B_{21} = g_1/g_2 B_{12}$ ,

$$\kappa_\nu \rho = h\nu \left( n_1 B_{12} - n_2 \frac{g_1}{g_2} B_{12} \right) = h\nu n_1 B_{12} \left( 1 - \frac{n_2}{n_1} \frac{g_1}{g_2} \right) \quad (5.180)$$

This leads us to the source function in terms of the Einstein coefficients

$$S_\nu \equiv \frac{j_\nu}{\kappa_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} \left( 1 - \frac{n_2}{n_1} \frac{g_1}{g_2} \right)} \quad (5.181)$$

This is the most general form of the source function for this simple system of two energy levels.

We can identify three behaviors. The first one is thermal radiation, for which the levels obey Boltzmann law, and  $n_1/n_2 = g_2/g_1 e^{-h\nu/kT}$

$$S_\nu = \frac{g_2 A_{21}}{g_1 B_{12}} \frac{1}{e^{h\nu/kT} - 1} \quad (5.182)$$

$$= \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad (5.183)$$

and the opacity is

$$\kappa_\nu \rho = h\nu n_1 B_{12} \left( 1 - e^{-h\nu/kT} \right) \quad (5.184)$$

The second is nonthermal radiation

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2} e^{h\nu/kT} \quad (5.185)$$

And the third occurs when

$$n_1 g_2 < n_2 g_1 \quad (5.186)$$

If  $n_2 \gg n_1$  there are too many atoms in the upper state. As there are not many absorptions, the intensity increases along the ray due to intense stimulated emission. This is called a *maser*.

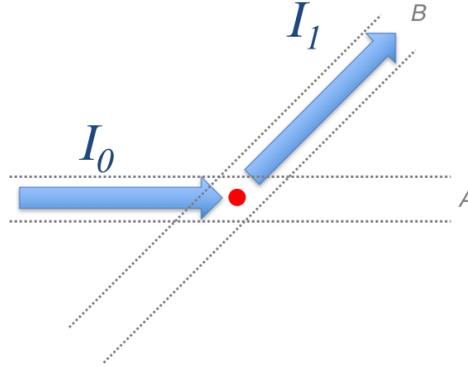


Figure 5.18: In an scattering event a photon is not destroyed, it simply changes direction. Yet, intensity is defined with an explicitly dependency on direction. For ray A, the initial intensity before the scattering event is  $I_0$ , and zero after the event. For ray A, an absorption event happened. Conversely, for ray B the initial intensity was zero, and after the scattering event it is  $I_1$ . For ray B, an emission event happened. We can parametrize scattering as independent absorption and emission events.

## 5.9 Scattering

Scattering is any process that changes the direction of a photon. Because we describe the radiation field in terms of intensity, which has an explicitly dependency on direction, a scattering event means that energy was removed from a beam and joined another beam. That is, one direction sees absorption, whereas another direction sees emission. Thus, in terms of intensity, we can mathematically describe a scattering event as two independent absorption and emission processes (Fig. 5.18).

Let us parametrize the absorption part in terms of a scattering coefficient  $\sigma_v$

$$dI_v = -\sigma_v \rho I_v \quad (5.187)$$

The scatterers in stars are mostly nonrelativistic electrons. For these, scattering is isotropic and coherent (no change in wavelength). The scattering is elastic, meaning that the ensemble of electrons do not gain energy or jump quantum states. Because scattering conserves energy, the photon that was lost from one direction will be found in another, so the total energy absorbed must be equal to the energy emitted

$$\oint \sigma_v \rho I_v d\omega = \oint j_v^{(s)} \rho d\omega \quad (5.188)$$

where  $j_v^{(s)}$  is a scattering emissivity. Since the scattering and emission are isotropic, we can remove them from the integral. The density is also considered isotropic, so we can write

$$\sigma_v \oint I_v d\omega = j_v^{(s)} \oint d\omega \quad (5.189)$$

and isolating  $j_\nu^{(s)}$

$$j_\nu^{(s)} = \sigma_\nu \frac{\oint I_\nu d\omega}{\oint d\omega} \quad (5.190)$$

we recognize the fraction of integrals as the mean intensity

$$j_\nu^{(s)} = \sigma_\nu J_\nu \quad (5.191)$$

For pure scattering,

$$\frac{dI_\nu}{ds} = -\sigma_\nu \rho (I_\nu - J_\nu) \quad (5.192)$$

Notice that for a blackbody ( $S_\nu \equiv B_\nu$ ), we can write the emission as

$$dI_\nu = B_\nu \kappa_\nu \rho dx \quad (5.193)$$

Considering thus absorption, emission, and scattering in radiative equilibrium

$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho (I_\nu - B_\nu) - \sigma_\nu \rho (I_\nu - J_\nu) \quad (5.194)$$

and grouping the negative terms and positive terms

$$\frac{dI_\nu}{ds} = -(\kappa_\nu + \sigma_\nu) \rho I_\nu + \rho(\kappa_\nu B_\nu + \sigma_\nu J_\nu) \quad (5.195)$$

comparing it to the equation of radiative transfer, we can define the effective opacity, or *extinction* coefficient

$$\kappa_\nu^{\text{eff}} \equiv \kappa_\nu + \sigma_\nu \quad (5.196)$$

likewise, the effective emissivity is

$$j_\nu^{\text{eff}} \equiv \kappa_\nu B_\nu + \sigma_\nu J_\nu \quad (5.197)$$

together, they define the effective source function

$$S_\nu^{\text{eff}} \equiv \frac{\kappa_\nu B_\nu + \sigma_\nu J_\nu}{\kappa_\nu + \sigma_\nu} \quad (5.198)$$

Notice that by defining

$$\omega_\nu \equiv \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} \quad (5.199)$$

the source function can be cast in the form

$$S_\nu^{\text{eff}} = \omega_\nu J_\nu + (1 - \omega_\nu) B_\nu \quad (5.200)$$

The quantity  $\omega_\nu$  relates scattering to absorption and is called *albedo*. It varies from pure absorption ( $\omega_\nu = 0$ ) to pure scattering ( $\omega_\nu = 1$ ). Finally, we can also define the effective optical depth as

$$\tau_\nu^{\text{eff}} = \int (\kappa_\nu + \sigma_\nu) \rho dx \quad (5.201)$$

## 5.10 Radiation Pressure

Radiation also carries momentum, so it exerts pressure over a surface. To get the momentum flux, recall that the momentum of a photon is  $E/c$ , in the direction of the beam  $\hat{n}'$

$$d\mathbf{p}_v = \frac{dE_v}{c} \hat{n}' = I_v dA \cdot d\omega dt dv \hat{n}' = \frac{I_v \cos \theta}{c} dA d\omega dt dv \hat{n}' \quad (5.202)$$

The pressure is found by finding the normal of this momentum to the area element, which is normal to  $\hat{n}$ .

$$dp_v^{\text{perp}} = d\mathbf{p}_v \cdot \hat{n} \quad (5.203)$$

$$= \frac{I_v \cos \theta}{c} dA d\omega dt dv \hat{n}' \cdot \hat{n} \quad (5.204)$$

$$= \frac{I_v \cos^2 \theta}{c} dA d\omega dt dv \quad (5.205)$$

And the pressure is the force over area,

$$dP_v \equiv \frac{dp_v^{\text{perp}}}{dt dA} \quad (5.206)$$

$$= \frac{I_v \cos^2 \theta}{c} d\omega dv \quad (5.207)$$

Integrating it over all directions, we get the total pressure

$$P_v = \frac{1}{c} \oint I_v \cos^2 \theta d\omega \quad (5.208)$$

## 5.11 The quantities $H_v$ and $K_v$

Similarly to the mean intensity, which is the directional average of the intensity, we can divide the flux and pressure by  $d\omega$  to write

$$J_v = \frac{1}{4\pi} \oint I_v d\omega \quad (5.209)$$

$$H_v = \frac{1}{4\pi} \oint I_v \cos \theta d\omega \quad (5.210)$$

$$K_v = \frac{1}{4\pi} \oint I_v \cos^2 \theta d\omega \quad (5.211)$$

Notice that  $H_v = F_v/4\pi$  and  $K_v = cP_v/4\pi$ , i.e., these quantities are related to intensity, flux, and pressure, respectively. These are just normalizations to remove factors of  $4\pi$  that sometimes appear in equations. The interesting fact to notice is that  $J$ ,  $H$ , and  $K$  are, respectively, the zeroth, first, and second moments<sup>3</sup> of the intensity with respect to the direction.

<sup>3</sup>A moment in mathematics is a combination of a physical quantity and a coordinate.

Quantity	unit	relation to SI
Length	centimeter (cm)	$10^{-2}$ m
Mass	gram (g)	$10^{-3}$ kg
Time	second (s)	same
Velocity	centimeter per second (cm/s)	$10^{-2}$ m/s
Acceleration	centimeter per second squared (cm/s <sup>2</sup> )	$10^{-2}$ m/s <sup>2</sup>
Force	dyne (g cm/s <sup>2</sup> )	$10^{-5}$ N
Energy	erg (g cm <sup>2</sup> /s <sup>2</sup> )	$10^{-7}$ J
Power	erg per second (erg/s)	$10^{-7}$ W
Pressure	bar (g cm <sup>-1</sup> s <sup>-2</sup> )	$10^{-1}$ Pa
kinematic viscosity	stokes (St $\equiv$ cm <sup>2</sup> /s)	$10^{-4}$ m <sup>2</sup> /s

Table 5.1: Selected units in the cgs system

Constant	symbol	value in cgs ( $\approx$ )	units
Speed of light	$c$	$3 \times 10^{10}$	cm s <sup>-1</sup>
Atomic mass unit	amu	$1.66 \times 10^{-24}$	g
electron mass	$m_e$	$9.1 \times 10^{-28}$	g
gravitational constant	$G$	$6.67 \times 10^{-8}$	cm <sup>3</sup> g <sup>-1</sup> s <sup>-2</sup>
electron charge	$e$	$4.8 \times 10^{-10}$	esu
Boltzmann constant	$k$	$1.38 \times 10^{-16}$	erg K <sup>-1</sup>
Stefan-Boltzmann constant	$\sigma$	$5.67 \times 10^{-5}$	erg s <sup>-1</sup> cm <sup>-2</sup> K <sup>-4</sup>
Planck constant	$h$	$6.62 \times 10^{-27}$	erg s
Radiation constant	$a$	$7.56 \times 10^{-15}$	erg cm <sup>-3</sup> K <sup>-4</sup>

Table 5.2: Selected constants in the cgs system

## 5.12 Problems

### Analytical problems

- Derive Eq. (5.14) from Eq. (5.13).
- Given that magnitude and flux are related by  $m = -2.5 \log_{10} F + C$ , what flux ratio is equivalent to the difference of one magnitude? Round the answer to three decimal points.
- Two stars, at a distance of 1.3 parsecs from the Sun, form a binary system with a mean separation of 20 AU, similar to that of Uranus from the Sun.
  - What is the projected separation of the stars measured from Earth, in arc-seconds?
  - The theoretical resolving power of a telescope, in radians, is  $1.2\lambda/D$ , where  $\lambda$  is the wavelength of light, and  $D$  (expressed in the same units as  $\lambda$ ) is the diameter of the objective lens or mirror. What is the minimum diameter for the objective of a telescope that can reveal the system as a visual binary, in centimeters? Assume  $5500\text{\AA}$  for the center of the visible spectrum. Knowing that the diameter of the eye's pupil is about 5 mm, can you resolve it with the naked eye?

- (c) Assuming that one star has magnitude 0 and the other one has magnitude 1, what is the magnitude you see with the naked eye? Warning: do not overlook the constant. Round the answer to two decimal points.
- (d) What are the stars' absolute magnitudes?
- (e) Knowing that the Sun's visual magnitude is -26.7, and assuming that the stars have equal bolometric corrections, what is the luminosity of each star, compared to the Sun's?
- (f) The Sun's effective temperature is 5777 K. Assuming that the stars have radii of  $1.2 R_{\odot}$  and  $0.8 R_{\odot}$ , respectively, what are their effective temperatures?
4. Show that, in general, the combined magnitude of a system of  $N$  objects is

$$M_{\text{total}} = -2.5 \log_{10} \sum_i^N 10^{-0.4m_i} \quad (5.212)$$

where  $m_i$  is the magnitude of each individual  $i$ -th object.

5. The Earth and the Moon are separated by an average distance of  $\approx 240$  thousand miles, or  $\approx 3.8 \times 10^{10}$  cm.
- (a) What is the maximum angular separation between the Earth and the Moon as seen in the Martian sky, measured in arcminutes? Mars is at  $\approx 1.4$  AU from the Sun.
- (b) If you stood on Mars, could you resolve the Earth-Moon system with the naked eye?
- (c) The Earth is seen from Mars with a magnitude of -2.5. The Moon is seen from Mars with magnitude 0.9. What is the magnitude an observer on Mars will measure if the Moon and the Earth are not resolvable? Round the answer to two decimal points.
6. The Andromeda galaxy is visible in the night sky as a diffuse cloud of 4th magnitude ( $V = 3.44$ , where  $V$  stands for magnitude in the visible waveband.)
- (a) Several ways to estimate the distance to Andromeda agree on the value of  $\approx 780$  kpc. Show that the absolute magnitude of the Andromeda galaxy in the same band is  $M_V \approx -21$ .
- (b) Assume that the radiation coming from Andromeda is all due to solar-like stars ( $M_{V,\odot} = 4.83$ ). How many stars are there in the Andromeda galaxy according to this approximation?
- (c) The actual number of stars in the Andromeda galaxy is in fact about a trillion. Based on that and your answer to the previous question, what can you conclude about the luminosity of the typical star in that galaxy?
- (d) The most common stars in the solar neighborhood are M dwarfs. Let us assume that the same holds true in other galaxies. M dwarfs range in mass from 0.08 to  $0.5 M_{\odot}$ , where  $M_{\odot}$  is the mass of the Sun. The mass-luminosity relationship for main sequence stars can be approximated as  $L/L_{\odot} = (M/M_{\odot})^{3.5}$ . With these new information, refine your estimate of the number of stars in the Andromeda galaxy. Assume that a characteristic M dwarf has a third of the mass of the Sun

7. The average person has  $1.5 \text{ m}^2$  of skin at a skin temperature of roughly  $300 \text{ K}$ . Consider the average person to be an ideal radiator standing in a room at a temperature of  $290 \text{ K}$ .
- Calculate the energy per second radiated by the average person in the form of blackbody radiation. Express the answer in watts.
  - Determine the peak wavelength  $\lambda_{\text{max}}$  of the blackbody radiation emitted by the average person. In what region of the electromagnetic spectrum is this wavelength found?
  - A blackbody also absorbs energy from its environment in this case from the  $290 \text{ K}$  room. The equation describing the absorption is the same as the equation for emission of blackbody radiation (Power=Area  $\times$  Flux). Calculate the energy per second absorbed by the average person, expressed in watts.
  - Calculate the net energy per second lost by the average person via blackbody radiation.
8. (a) Show that in the limit of low energy ( $h\nu \ll kT$ ), the Planck spectrum becomes  $B_\nu(T) = 2kTv^2/c^2$ . (note: this is called *Rayleigh-Jeans law*), and was derived before Planck's law, via classical arguments. Because the emission increases unboundedly for short wavelengths, this law lead to what was called "ultraviolet catastrophe". Yet, it agrees well with the experiments for long wavelengths. We talk of a "Rayleigh-Jeans tail" in the radio spectrum).
- (b) Show that in the limit of high energy ( $h\nu \gg kT$ ), the Planck spectrum becomes

$$B_\nu(T) = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}. \quad (5.213)$$

This limit is called *Wien's law* (not to be confused with Wien's displacement law), and it agrees well with the experiments for short wavelengths.

9. Wien's displacement law relates the peak frequency  $\nu_{\text{max}}$  of the blackbody radiation curve and the temperature  $T$  of the same blackbody. Curiously, the peak wavelength  $\lambda_{\text{max}}$  is not obtained by  $\lambda_{\text{max}} = c/\nu_{\text{max}}$  as one would naively expect. Explain why not and derive the equivalent of Wien's displacement law for wavelength (Eq. 5.71).
10. Consider that the human body is at  $\approx 300 \text{ K}$ .
- Evaluate the flux of blackbody photons inside your eye.
  - The area of the eye's pupil is about  $0.1 \text{ cm}^2$ . What is the luminosity leaving your eye from the pupil?
  - Compare this with the luminosity that enters your eye while looking at a  $100 \text{ W}$  light bulb that is  $1 \text{ m}$  away. You can assume that the light bulb is  $100\%$  efficient, although in reality it converts only a few percent of its  $100$  watts into visible photons.
  - Why is it dark when you close your eyes?

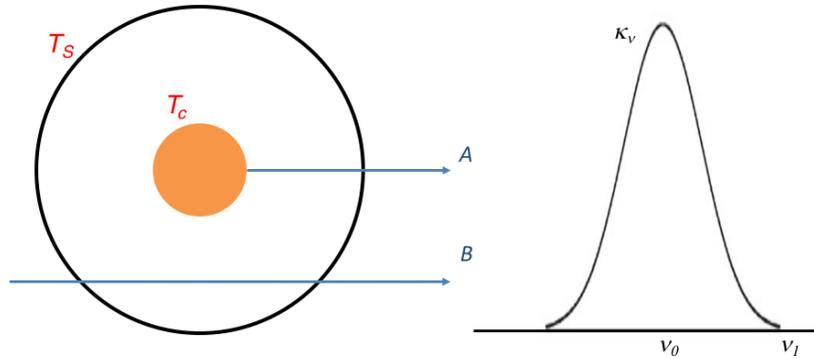


Figure 5.19: Blackbody emitter at temperature  $T_c$  surrounded by an absorbing shell at temperature  $T_s$ , viewed along rays A and B. Right: Absorption coefficient of the material in the shell.

11. Find an expression for  $n_\lambda d\lambda$ , the number density of blackbody photons (the number of blackbody photons per  $\text{cm}^3$  with a wavelength between  $\lambda$  and  $\lambda + d\lambda$ ). Find the total number of photons inside a kitchen oven set at 480 K, assuming a volume of 500 liters (1 liter =  $10^3 \text{ cm}^3$ ).
12. A circular disk of radius  $R$  emits radiation of intensity  $I_\nu$ , in all directions. For a distance  $r$ , (a) describe the radiation field, (b) obtain the mean intensity and (c) the flux if the observer is located in the axis of symmetry of the disk.
13. A source emits as a blackbody at temperature  $T_c$ . The source is enclosed by another source, a hollow shell, emitting also as a blackbody at a temperature  $T_s$  ( $T_s < T_c$ ). The shell is mostly optically thin, absorbing in a narrow spectral line: its opacity is relatively large at frequency  $\nu_0$  but mostly transparent at other frequencies, such as  $\nu_1$ :  $\kappa_{\nu_0} \gg \kappa_{\nu_1}$  (see Fig. 5.19). The object is observed at frequencies  $\nu_0$  and  $\nu_1$  and along two rays A and B shown in Fig. 5.19. Based on the equation of radiative transfer (in the optically thin limit), answer *mathematically* the questions below
  - (a) At which frequency will the observed intensity be larger when observed along ray A? Along ray B?
  - (b) Answer the preceding question if  $T_s > T_c$ .
14. Show that for a cloud of optical depth  $\tau = 1$  and diameter  $L$ , back-illuminated by a source of intensity  $I_0$ , the intensity measured by an observer is

$$I = (I_0 - \chi L)e^{-1} + \chi L \quad (5.214)$$

where  $\chi = j\rho$ .

15. Show that the interaction time between a free electron and a free proton in an ensemble of number density  $n$  and temperature  $T$  is

$$t \approx 10^{-6} \text{s} \left( \frac{n}{\text{cm}^{-3}} \right)^{-1/3} \left( \frac{T}{\text{K}} \right)^{-1/2} \quad (5.215)$$

*note:* in the above relation,  $K$  is the kelvin unit of temperature, not the Boltzmann constant.

16. Radiation is received from the halo of a galaxy. The temperature of the source is estimated as  $T = 10^6$  K. The mass  $M$  and radius  $R$  of the source are estimated to be  $M = 10^{11} M_\odot$  and  $R = 10$  kpc. The radiation is thought to be Bremsstrahlung.

- (a) Test this assumption by checking if the cloud is optically thin. For that, assume that the cloud is entirely made of ionized hydrogen, and that the photon mean free path is

$$\bar{l} = \frac{1}{n_e \sigma_T} \quad (5.216)$$

where  $n_e$  is the electron density and the cross section  $\sigma_T = 6.7 \times 10^{-25}$  cm<sup>2</sup>.

- (b) What electron density is required to make the cloud optically thick?  
 (c) Compute the cloud luminosity knowing that the (frequency-integrated and directionally-averaged) Bremsstrahlung emissivity can be approximated as

$$\chi = j\rho = 1.68 \times 10^{-27} T^{1/2} n^2 \text{ erg s}^{-1} \text{ cm}^{-3} \quad (5.217)$$

- (d) As the cloud radiates, it cools down. The average kinetic energy of a thermal electron is  $3kT/2$ . What is the cooling time of the cloud, in millions of years? (Hint: Remember that luminosity is energy *lost*, and construct from the energy and the luminosity a quantity that has unit of time.)

17. In deriving the absorption and emission coefficients, Einstein originally used only  $A_{21}$  and  $B_{12}$ , i.e., the coefficients for spontaneous emission and radiative absorption. With these, he could only derive Wien's law. To recover the Planck function, Einstein had to postulate the existence of stimulated emission, the coefficient  $B_{21}$ . Show that when considering only  $A_{21}$  and  $B_{12}$  one indeed gets Wien's law.

18. Neutrinos have nearly negligible mass, traveling at about the speed of light. For some astrophysical purposes, they can be considered as radiation. We have been studying radiative transfer assuming that the radiation field is composed of photons, but many of the same properties would apply to a radiation field composed of neutrinos. One of the main differences between photons and neutrinos is that the former are bosons, whereas the latter are fermions and obey Pauli's exclusion principle. The presence of a neutrino in the radiation field will prevent the emission of another in the same state. So, instead of stimulated emission, neutrinos experience inhibited emission: not all neutrinos that can be emitted are emitted, and the amount of blocked neutrinos is proportional to the intensity of the neutrino radiation field.

Given these information, and assuming that the Einstein coefficients are the same as in the photon case, show that an equilibrium neutrino field will have intensity given by

$$I_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) + 1} \quad (5.218)$$

19. A free neutron is observed to decay in nature with a half-life of 15 minutes, converting into a proton, an electron, and a neutrino. Yet, a neutron is stable inside the nucleus. Can you explain why? Hint: if the neutron decays in a nucleus, where will the proton end up?
20. Photons are produced in a cloud of radius  $R$  at the uniform and isotropic rate  $\Gamma$  (energy per unit volume per unit time). Neglect absorption of these photons, i.e., the medium is optically thin.

- (a) Assume that the source is completely resolved. Show that the intensity toward the center of the cloud is

$$I_0 = \frac{\Gamma R}{2\pi} \quad (5.219)$$

- (b) The cloud is at a distance  $d$  away. Show that the flux is

$$F = \frac{\Gamma R^3}{3d^2} \quad (5.220)$$

- (c) A detector on Earth has angular resolution  $\theta_d$ , meaning the smallest angular area in the sky it can measure is  $\pi\theta_d^2/4$ . We call this the detector's *beam*. A crude way to measure the intensity given the flux is  $I = F \text{ beam}^{-1}$ .

We showed that the intensity does not depend on distance. In fact this is only true if the source is resolved. An unresolved source has angular area smaller than the beam, and we cannot define different directions (different solid angles). Show that in this case the intensity decreases according to

$$I = \frac{2I_0}{3} \left( \frac{\theta_s}{\theta_d} \right)^2 \quad (5.221)$$

where  $I_0$  is the resolved intensity and  $\theta_s$  the angular size of the source.

21. The flux of a star is reduced by absorption from the atmosphere:  $I = I_0 e^{-\tau}$ , with  $\tau = A \int \rho(z)\kappa(z)dz$ , with  $A$  the airmass,  $\kappa$  the opacity and  $z$  the altitude. The airmass is given by  $1/\cos(\theta)$ , with  $\theta$  the zenith angle. The optical depth ( $\tau$ ) is in practice difficult to calculate as  $\rho(z)$  and  $\kappa(z)$  are not precisely known. Show that if we do two measurements of the received flux ( $I_1, I_2$ ) at different Airmasses ( $A_1, A_2$ ) we can find  $I_0$  (assuming the properties of the atmosphere do not change with time).

22. *Limb darkening.*

Consider the spherical cloud in Fig. 5.20. It has radius  $R$  and emits isotropically with a uniform source function  $S_\nu$ .

- (a) Show geometrically that, at a projected distance  $b$  from the cloud center (Fig. 5.20), the chord parallel to the diameter has length  $L$  given by

$$L = 2\sqrt{R^2 - b^2} \quad (5.222)$$

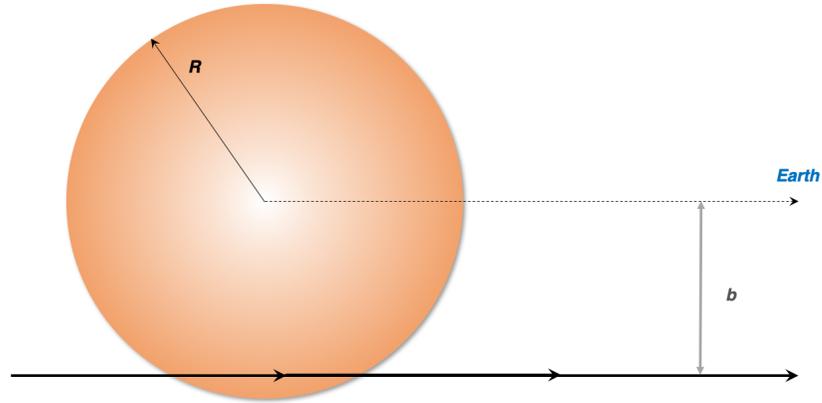
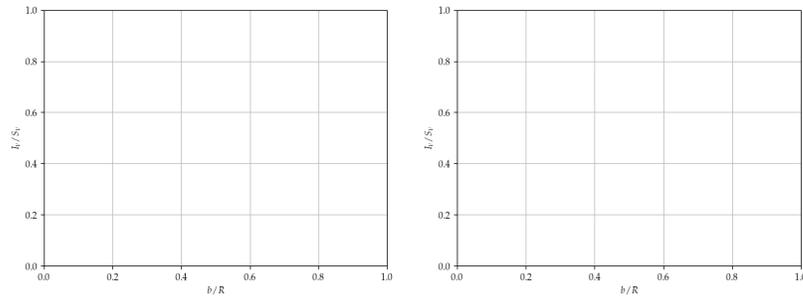


Figure 5.20: Viewing geometry.

Figure 5.21: Plot in the left panel the solution for  $\tau_0 = 1$  and on the right panel the solution for  $\tau_0 = 10$ .

- (b) If the optical depth through the diameter of the cloud is  $\tau_0$ , show that the intensity  $I_v$  of the sphere as a function of projected radius  $b$ , as viewed from a long distance away, is

$$I_v = S_v \left( 1 - e^{-\tau_0 \sqrt{1-(b/R)^2}} \right) \quad (5.223)$$

- (c) Assuming  $\tau_0 = 1$ , calculate  $I_v/S_v$  for  $b/R = [0.0, 0.2, 0.4, 0.6, 0.8, 0.9, 1.0]$  and plot it in the grid shown in Fig. 5.21.  
 (d) Repeat the above for  $\tau_0 = 10$ .

23. *Limb brightening.* Consider a spherical shell of radius  $R$  and thickness  $\Delta R \ll R$  Fig. 5.22 that emits isotropically and with a uniform source function  $S_v$ .

- (a) If the optical depth through the diameter of the shell is  $\tau_0 \ll 1$ , show the intensity  $I_v$  as a function of projected radius, as viewed from a long distance

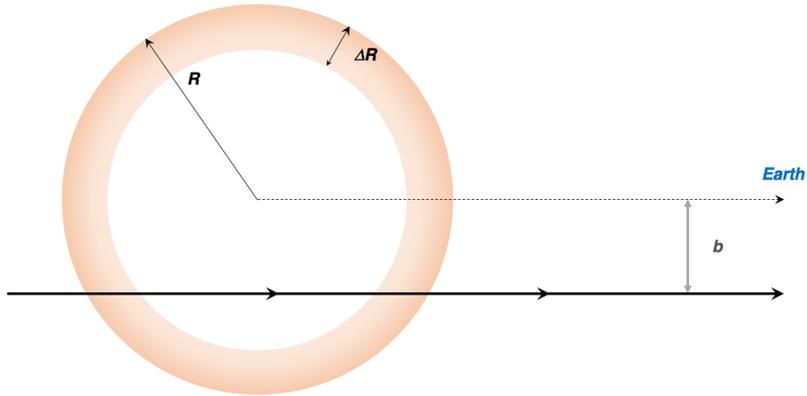


Figure 5.22: Viewing geometry.

away, is

$$I_\nu = S_\nu \frac{\tau_0}{\sqrt{1 - (b/R)^2}} \quad (5.224)$$

- (b) Assuming  $\tau_0 = 0.1$ , calculate  $I_\nu/S_\nu$  for  $b/R = [0.0, 0.2, 0.4, 0.6, 0.8, 0.9, 1.0]$  and plot it in the grid shown in Fig. 5.23.
- (c) The above solution implies that the intensity is infinite when  $b = R$ . This is obviously a feature of the approximation  $\Delta R \ll R$ . Show that when the thickness of the shell is finite, the intensity peaks at

$$I_\nu = S_\nu \tau_0 \frac{R}{\Delta R} \sqrt{1 - \left(1 - \frac{\Delta R^2}{R^2}\right)} \quad (5.225)$$

$$\approx S_\nu \tau_0 \sqrt{\frac{2R}{\Delta R}} \quad (5.226)$$

This is a way to measure the thickness of expanding shells.

24. Consider again the cloud in Fig. 5.20. Its emission is parametrized in terms of specific luminosity density  $\Gamma_\nu$  (luminosity per unit volume and per unit frequency).

- (a) Through dimensional analysis, relate this parameter to emissivity  $j_\nu$ , showing that

$$j_\nu \rho = \frac{\Gamma_\nu}{4\pi} \quad (5.227)$$

- (b) Defining the bolometric luminosity density  $\Gamma = \int_0^\infty \Gamma_\nu d\nu$ , find the luminosity and the effective temperature of the cloud.
- (c) What is the specific flux  $F_\nu$  measured from Earth?

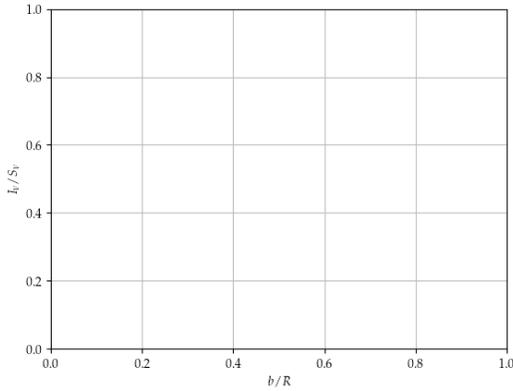


Figure 5.23: .

### Numerical problems

1. In deriving Wien's displacement law, we encountered an equation that had no analytical solution. This was

$$f(x) = xe^x - 3e^x + 3 \quad (5.228)$$

One of the roots is  $x = 0$ . In a computer language of your choice (python is recommended), write a numerical code to find the other root of this function, with precision of 4 decimal points.

2. In the course webpage you will find a stellar model, in the form a table containing values of temperature, density, and opacity at various points near the surface of the star. In this problem, you will use these values to calculate the optical depth. The first point listed is at the surface of the stellar model.
  - (a) Find the optical depth at each point by numerically integrating  $d\tau = \kappa\rho ds$ .
  - (b) Make a graph of the temperature vs the optical depth
  - (c) Eddington provided a useful expression for the temperature in the gray approximation:  $T^4 = {}^3/4T_{\text{eff}}^4(\tau + {}^2/3)$ . Plot these values of  $T$  on the same graph.
  - (d) The model used utilizes a simplifying assumption that the surface temperature is zero. Comment on the validity of the surface value of  $T$  that you found.
3. An astrophysical source is measured to have magnitude  $12 \pm 0.5$  in the  $L$  photometric band (centered at  $3.5\mu\text{m}$ ), and  $20 \pm 0.5$  in the  $H$  band (centered at  $1.65\mu\text{m}$ ).
  - (a) Plot the data in a diagram of wavelength vs magnitude (with error bars). Label the axes appropriately.

- (b) Assume that the source is emitting like a blackbody, plot the blackbody curve that best fits the data. What is the temperature of the source? Label the curve appropriately.
- (c) What is the error of the temperature you determined?
4. The blackbody emissivity of a source emitting thermal bremsstrahlung is

$$j_\nu = \frac{8}{3} \sqrt{\frac{2\pi}{3}} \frac{e^6}{m_e^2 c^3} \left(\frac{m_e}{kT}\right)^{1/2} \exp\left(-\frac{h\nu}{kT}\right) n_e n_p g_{\text{ff}} \quad (5.229)$$

Here  $e$  is the electron charge,  $m_e$  is the electron mass,  $c$  is the speed of light,  $k$  is Boltzmann's constant,  $T$  is the temperature,  $h$  is Planck's constant;  $n_e$  and  $n_p$  are the number densities of electrons and protons, respectively. The quantity  $g_{\text{ff}}$  is called Gaunt factor and represents a quantum mechanical correction at low frequencies (our treatment so far has been purely classical). An approximation for it is

$$g_{\text{ff}} \approx \ln \left\{ \exp \left[ 5.960 - \frac{\sqrt{3}}{\pi} \ln \left( \left( \frac{\nu}{10^9 \text{ Hz}} \right) \left( \frac{T}{10^4 \text{ K}} \right)^{-3/2} \right) \right] + e \right\} \quad (5.230)$$

Assume that the body is in thermal equilibrium so you can apply Kirchhoff law and the opacity is

$$\alpha_\nu = \frac{j_\nu}{B_\nu} \quad (5.231)$$

Consider that the source has a radius of  $R = 10^{15}$  cm, temperature of  $T = 10^7$  and number densities  $n_e = n_p = 10^{10} \text{ cm}^{-3}$ . Plot

- The emissivity vs frequency.
- The opacity vs frequency.
- The optical depth vs frequency.
- The intensity vs frequency.

For finding the intensity, assume that the emission is optically thin until  $\tau = 1$  and only then absorption becomes important. For all plots label the axes appropriately and use logarithmic axes when convenient.

- When graphing the intensity, overplot the blackbody curve. Keep all parameters fixed and vary the density. To order of magnitude accuracy, what is the density at which the emission become blackbody?
- Repeat the exercise above for a range of temperatures from  $T = 100\text{K}$  to  $T = 10^7\text{K}$ , finding at each temperature the density  $\rho_{bb}$  where emission becomes blackbody. Graph  $T$  vs  $\rho_{bb}$ .
- Scattering is defined as a process by which a photon is bounced off into a different direction. This is different from true absorption, whereby a photon ceases to exist. Whereas a single scattering event can be treated as absorption in one direction and emission in another, multiple random scattering events increase

the likelihood that a photon will be put back into the original beam. Thus, we expect that multiple scattering events will lead to less loss of intensity than true absorption would. In fact, the transmission  $T$  — defined as the ratio between the transmitted intensity  $I(\tau)$  to original intensity  $I_0$  — after multiple pure scattering events can be calculated to be

$$T \equiv \frac{I(\tau)}{I_0} = \frac{2}{2 + \tau} \quad (5.232)$$

which reveals much less extinction than the exponential scaling for pure absorption (otherwise cloudy days would be really dark). Let us derive this result via an idealized thought experiment.

Imagine a cloud layer as a slab of thickness  $L$ . Sunlight of intensity  $I_0$  is shining through it. The cloud is composed of droplets, of number density  $\nu$ , each of radius  $r$ . The droplets do not absorb the photons in the sunlight, but simply bounce them around. Each scattering event throws the photon in a random direction. Because the cloud layer is much larger than it is thick, we can consider only two directions, up and down. After a random walk, some photons will traverse the cloud deck and be thrown down towards the surface. Conversely, some will find themselves bounced upwards and sent back to space.

- (a) Show that the optical depth is

$$\tau = \nu\pi r^2 L \quad (5.233)$$

- (b) Find the number density of photons in the incoming radiation. Remember the relation between mean intensity and energy density.  
 (c) The random walk that the photons execute upon entering the cloud deck can be described by a diffusion equation. So, the number density of photons in the cloud will follow

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial z^2} \quad (5.234)$$

where  $D$  is a diffusion coefficient (its unit should be  $\text{cm}^2/\text{s}$ ). Show that

$$D = \frac{c}{\nu\pi r^2} \quad (5.235)$$

- (d) Solve the diffusion equation for the steady-state case,  $\partial_t = 0$ . There should be two constants of integration.  
 (e) You need two boundary conditions to find the two constants of integration. One of them is at the bottom of the cloud

$$n_t = n(z = L) \quad (5.236)$$

where the subscript  $t$  stands for “transmitted”. These photons are the transmitted flux. The second boundary condition is that the flux at the upper part of the cloud, at  $z = 0$ , must equal the incoming flux  $F_i$  minus the outgoing, reflected flux  $F_r$ , so  $F(z = 0) = F_i - F_r$ .

Another constrain is that a diffusion flux must obey

$$F = -D \frac{\partial n}{\partial z} \quad (5.237)$$

Lastly, the flux should obey conservation of energy, so that  $F_i = F_r + F_t$ , i.e., the incident flux must equal the reflected plus transmitted. Use these to find the transmission ratio  $T$  in terms of  $\tau$ .

- (f) Plot in the same graph the transmission from multiple scattering and the transmission from pure absorption down to optical depth 10.
8. In this exercise you will write a code that shows the behavior described in the previous exercise. For simplicity, assume one direction only. Define a line from 0 to 1, with a scatterer at every 0.1 interval. Send a photon along the way. At every encounter with a scatterer it has a probability of 0.5 to either forward or backscatter. Follow the photon until it is either sent to  $x < 0$ , where it becomes part of the reflected flux, or until it escapes to  $x > 1$ , where it becomes part of the transmitted flux.
- (a) Send 10 such photons and count how many get transmitted. Repeat it for 100, 1000, 10 000, 100 000, and  $10^6$  photons. Plot the photon number versus transmission ratio (axes in log).
- (b) What is the scattering optical depth of the medium?
- (c) Modify the program and assume that instead of forward or backscattering, the photon has 0.5 probability of being absorbed during an encounter. Redo the exercise with 10, 100, 1000, 10 000, 100 000 and  $10^6$  photons. What is the transmission rate?
- (d) Record the number density of photons  $n(x)$  along the line for the case of  $10^6$  photons for the absorption case and plot  $n(x)$  vs  $x$ . Do you see the exponential absorption law? What is the absorption opacity of the medium?
- (e) Why are the scattering and absorption results so different?