

Chapter 6

Stellar Physics: Atmospheres, Structure, and Evolution.

We have studied and mastered the equations of radiative transfer. We have the deck of cards and we know the rules. It is time to play the game. Let us apply the equations of radiative transfer to the case of a star and understand how radiation is transported through a stellar atmosphere.

6.1 Stellar Atmospheres

6.1.1 The equation of Radiative Transfer in spherical coordinates

Since radiation travels in straight lines, we have not had to care about coordinate systems so far. Yet, in the case of star, the spherical geometry naturally calls for spherical coordinates. Our first step towards a description of stellar atmospheres is thus to write the radiative transfer equation in spherical coordinates.

Because of the spherical symmetry of the problem, the depth is the coordinate where optical depth increases, and without loss of generality we can align that with the z -axis. The equation of radiative transfer involves a derivative with respect to the optical depth, so along the z coordinate. We need to write this derivative in terms of spherical coordinates.

The transformations from Cartesian to spherical derivatives are

$$\begin{bmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{bmatrix} = \mathbf{A} \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{bmatrix} \quad (6.1)$$

where \mathbf{A} is the Jacobian matrix, given by

$$\mathbf{A} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{bmatrix}. \quad (6.2)$$

We can calculate the elements of the Jacobian matrix, given

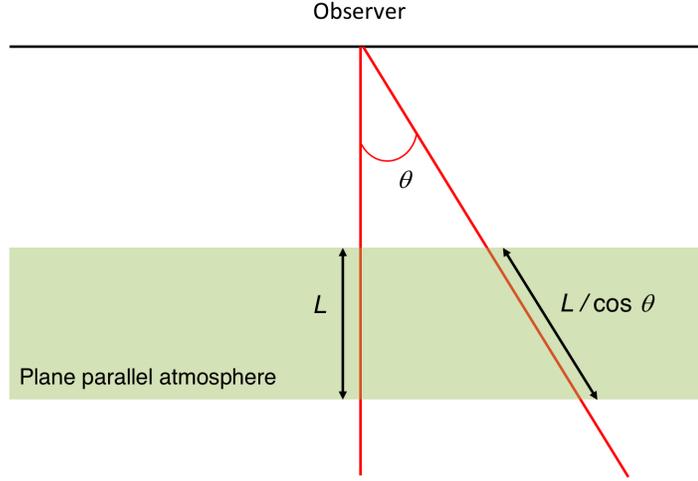


Figure 6.1: Light rays coming from different lines of sight had to traverse different lengths within a plane-parallel atmosphere of given thickness.

$$x = r \sin \theta \cos \phi, \quad (6.3)$$

$$y = r \sin \theta \sin \phi, \quad (6.4)$$

$$z = r \cos \theta, \quad (6.5)$$

resulting in

$$\mathbf{A} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & -r \sin \theta \cos \phi & 0 \end{bmatrix}, \quad (6.6)$$

for which the inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} \sin \theta \cos \phi & r^{-1} \cos \theta \cos \phi & -\frac{\sin \phi}{r \sin \theta} \\ \sin \theta \sin \phi & r^{-1} \cos \theta \sin \phi & \frac{\cos \phi}{r \sin \theta} \\ \cos \theta & -r^{-1} \sin \theta & 0 \end{bmatrix}. \quad (6.7)$$

Combining Eq. (6.1) and Eq. (6.7), the derivative in z is

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}. \quad (6.8)$$

Substituting Eq. (6.8) in the Radiative Transfer equation, we arrive at its form in spherical coordinates

$$\frac{\cos \theta}{\kappa_v \rho} \frac{\partial I_v}{\partial r} - \frac{\sin \theta}{\kappa_v \rho r} \frac{\partial I_v}{\partial \theta} = S_v - I_v \quad (6.9)$$

6.1.2 Plane-parallel approximation

Since the length of the atmosphere of a star is small compared to the radius, we can assume $r \gg 1$, ignoring the second term in Eq. (6.9). The equation becomes

$$\frac{\cos \theta}{\kappa_v \rho} \frac{\partial I_v}{\partial r} = S_v - I_v \quad (6.10)$$

This is equivalent to ignoring the curvature of the star, which names this trick the *plane-parallel* approximation.

Notice also that we can absorb the co-sine in the optical depth, writing $\tau_v \rightarrow \tau_v / \cos \theta$ and the equation would have the same form as we have so far studied. This is a geometrical construct: the optical depth is defined along the ray. For a plane-parallel atmosphere of depth L , light rays propagating obliquely by a angle θ with the vertical will traverse a longer path, $L / \cos \theta$ Fig. 6.1 .

An important mathematical change, though, is how we define the direction of optical depth increase. Photons are traveling upward, but we measure optical depth in terms of material in our line of sight. From our perspective, from outside the star, the optical depth in the atmosphere increases with depth, opposite to the movement of the photons.

If optical depth and geometrical depth increase in opposite directions, the infinitesimals must have opposite sign, leading us to re-define

$$d\tau = -\kappa \rho dr \quad (6.11)$$

Going to greater τ we are encountering exponentially more intensity. The reader should keep in mind that the sign reversal simply means this change of perspective.

We arrive thus at the form of the equation of radiative transfer usually used in the plane-parallel approximation

$$\cos \theta \frac{\partial I_v}{\partial \tau_v} = I_v - S_v \quad (6.12)$$

6.1.3 Radiative equilibrium

The atmosphere of a star simply transports all energy that flows through it, produced in the core. Without sources and sinks, the energy is rigorously conserved

$$\frac{dE}{dt} = 0 \quad (6.13)$$

We can relate the above equation to the flux using the definition of flux $dE \equiv F dA dt$. Dividing both sides by dt ,

$$\frac{dE}{dt} = \mathbf{F} \cdot d\mathbf{A} \quad (6.14)$$

integrating over the whole area of the star and using Gauss theorem, the RHS becomes

$$\oint \mathbf{F} \cdot d\mathbf{A} = \int \nabla \cdot \mathbf{F} dV \quad (6.15)$$

Since $dE/dt = 0$, the integrand must be zero for all volume elements. So,

$$\boxed{\nabla \cdot \mathbf{F} = 0} \quad (6.16)$$

For the plane-parallel approximation, this simplifies to $dF/dz = 0$, or $F \equiv \text{const}$. That is, *the bolometric flux is constant*. We can define it in terms of the luminosity and radius of the star

$$F_0 = \frac{L}{4\pi r^2} \quad (6.17)$$

We can also use Stefan-Boltzmann's law $F = \sigma T^4$ to define an *effective temperature*

$$F_0 = \sigma T_{\text{eff}}^4 \quad (6.18)$$

the effective temperature is the temperature of a blackbody that emits the same bolometric flux. *Even though we cannot always approximate an emitter by a blackbody, we can always define an effective temperature.*

Having found a solution for the flux, we can use Eq. (6.12) to further constrain the problem. Integrating the equation in solid angle

$$\frac{d}{d\tau_\nu} \oint I_\nu \cos \theta d\omega = \oint I_\nu d\omega - \oint S_\nu d\omega \quad (6.19)$$

since we are in radiative equilibrium, we can assume S_ν to be isotropic

$$\frac{dF_\nu}{dz} = 4\pi\kappa_\nu\rho(J_\nu - S_\nu). \quad (6.20)$$

Integrating the above equation in frequency

$$\frac{d}{dz} \int_0^\infty F_\nu d\nu = 4\pi \int_0^\infty \rho (J_\nu - S_\nu) d\nu \quad (6.21)$$

The LHS is $\nabla \cdot \mathbf{F}$, which in radiative equilibrium is zero. Therefore, the RHS is also zero. So, we find the result

$$\boxed{\int_0^\infty \kappa_\nu J_\nu d\nu = \int_0^\infty \kappa_\nu S_\nu d\nu} \quad (6.22)$$

or, recalling $S_\nu = j_\nu/\kappa_\nu$

$$\int_0^\infty \kappa_\nu J_\nu d\nu = \int_0^\infty j_\nu d\nu. \quad (6.23)$$

This is an interesting result. The RHS is the direction-integrated bolometric emission. The LHS is the direction-integrated bolometric absorption. What is emitted in a wavelength is absorbed in another. Eq. (6.23) is a statement of conservation of energy.

Lastly, we look at the pressure equation. In radiative equilibrium, the condition for pressure can be found by multiplying the RT equation by $\cos\theta$, and integrating in angle.

$$\frac{d}{d\tau_\nu} \oint I_\nu \cos^2 \theta d\omega = \oint I_\nu \cos \theta d\omega - S_\nu \oint \cos \theta d\omega \quad (6.24)$$

The integral in the LHS is $4\pi K_\nu$, the first in the RHS is the flux, and the second is zero. So,

$$\frac{dK_\nu}{d\tau_\nu} = H_\nu \quad (6.25)$$

Or, in terms of physical quantities,

$$\boxed{\frac{dP_\nu}{d\tau_\nu} = \frac{F_\nu}{c}} \quad (6.26)$$

Again this is an interesting result. The LHS is a pressure gradient. A force. Eq. (6.26) states that this force arises directly from the radiation flux.

6.1.4 Grey approximation

The above equations describe an atmosphere in radiative equilibrium, yet the dependencies in frequencies render the problem formidable to solve. In practice, we are dealing with infinite equations, one for each monochromatic frequency. A very stringent yet sometimes useful approach is the *grey approximation*, which assumes that the opacity does not depend on wavelength. Under these very limiting conditions, we can simply integrate the RT equation in frequency, to find

$$\cos\theta \frac{dI}{d\tau} = I - S \quad (6.27)$$

With that approximation, the equations of radiative equilibrium, Eq. (6.16), Eq. (6.23), and Eq. (6.26) reduce to

$$\boxed{F = F_0} \quad (6.28)$$

$$\boxed{J = S} \quad (6.29)$$

$$\boxed{P = \frac{F_0}{c}\tau + \text{const}} \quad (6.30)$$

In the grey atmosphere, the source function is simply the mean intensity. These equations are so simple that it makes sense to look for situations where an atmosphere can be treated in the grey approximation.

Mostly, the frequency dependency lies in the opacity κ_ν , which, by Kirchhoff's first law, Eq. (5.144)), translates into emission. To find a grey description, we need to find suitable averaging of the opacity in frequency. Two such averages are the *Rossland mean* and the *Planck mean* opacities.

6.1.4.1 Rossland mean opacity

The Rossland approximation is one such use of the grey approximation. In the deep interior the medium becomes optically thick and, according to Kirchhoff's 1st law, the intensity should approach the Planckian. We can therefore expand the intensity in Taylor series around a generic optically thick depth z_0

$$I_\nu(z) \approx B_\nu + \frac{dB_\nu}{dz}(z - z_0) + \mathcal{O}(z - z_0)^2. \quad (6.31)$$

Now, using the definition of optical depth

$$d\tau_\nu = -\kappa_\nu \rho dz / \cos\theta, \quad (6.32)$$

and dropping the second order term in Eq. (6.31), we have

$$I_\nu \approx B_\nu - \frac{\cos\theta}{\kappa_\nu \rho} \frac{\partial B_\nu}{\partial z}. \quad (6.33)$$

We now use the definition of flux

$$F_\nu = \oint I_\nu \cos \theta d\omega \quad (6.34)$$

$$= \int_0^{2\pi} \int_0^\pi I_\nu \cos \theta \sin \theta d\theta d\phi \quad (6.35)$$

Assuming azimuthal symmetry and substituting $\mu = \cos \theta$, we write

$$F_\nu = 2\pi \int_{-1}^1 I_\nu \mu d\mu \quad (6.36)$$

Substituting Eq. (6.33),

$$F_\nu = 2\pi B_\nu \int_{-1}^1 \mu d\mu - \frac{2\pi}{\kappa_\nu \rho} \frac{\partial B_\nu}{\partial z} \int_{-1}^1 \mu^2 d\mu. \quad (6.37)$$

The first term integrates to zero, leaving only the second term, which integrates to 2/3, i.e.,

$$F_\nu = -\frac{4\pi}{3\kappa_\nu \rho} \frac{\partial B_\nu}{\partial z} \quad (6.38)$$

$$(6.39)$$

Since the Planckian does not depend on depth, but on temperature, we can use the chain rule to replace it by a temperature gradient

$$\frac{\partial B_\nu}{\partial z} = \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial z}. \quad (6.40)$$

The monochromatic flux is then

$$F_\nu = -\frac{4\pi}{3\kappa_\nu \rho} \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial z} \quad (6.41)$$

We can integrate it in frequency to find

$$F = \int_0^\infty F_\nu d\nu = -\frac{4\pi}{3\rho} \frac{\partial T}{\partial z} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu \quad (6.42)$$

To bypass the frequency integration, we can define the **Rossland mean opacity**

$$\boxed{\frac{1}{\kappa_R} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}} \quad (6.43)$$

which is a weighted mean. Although the numerator of the Rossland mean opacity is unknown, because of the dependency on opacity, the denominator can be evaluated

$$\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial}{\partial T} \frac{\sigma}{\pi} T^4 = \frac{4\sigma}{\pi} T^3 \quad (6.44)$$

Substituting Eq. (6.43) and Eq. (6.44) into Eq. (6.42), we finally find the flux as a function of depth in the radiative atmosphere in the Rossland approximation

$$F(z) = -\frac{16}{3} \frac{\sigma T^3}{\kappa_R \rho} \frac{\partial T}{\partial z}. \quad (6.45)$$

The Rossland approximation is a very good approximation for the optically thick case, where $\tau_\nu \gg 1$. It is valid whenever the radiation field is isotropic over distances comparable to or less than a radiation mean free path, such as in local thermal equilibrium. Notice that the Rossland mean opacity is a weighted average of κ_ν^{-1} so that frequencies at which the opacity is small tend to dominate the flux: the Rossland mean opacity controls the transport of radiation.

6.1.4.2 Rossland vs Planck mean opacities

The Planck mean opacity is defined as

$$\kappa_P \equiv \frac{\int_0^\infty \kappa_\nu B_\nu d\nu}{\int_0^\infty B_\nu d\nu} \quad (6.46)$$

In contrast to the Rossland mean opacity, that favors transparent wavelengths and thus controls the radiation flux, the Planck opacity is a weighted average that favors high opacities. As such, the Planck opacity is the opacity of choice for describing processes such as absorption and emission.

6.1.5 The Eddington solution

An ingenious solution to the gray case in the plane-parallel approximation was presented by Sir Arthur Eddington. Eddington assumed that the intensity could be decomposed into the contribution from two directions: up and down (Fig. 6.2). In this two-ray approximation, we have I_+ as the intensity directed outwards (from the stellar interior toward the surface), and I_- as the intensity directed inwards (from the surface to the interior). That is, the radiation field is

$$I = \begin{cases} I_+ & \text{for } 0 \leq \theta < \pi/2 \quad \text{and} \quad 0 \leq \phi \leq 2\pi \\ I_- & \text{for } \pi/2 \leq \theta < \pi \quad \text{and} \quad 0 \leq \phi \leq 2\pi \end{cases} \quad (6.47)$$

We can then express the mean intensity J , flux F , and pressure P in terms of this intensity field. The mean intensity J is

$$J = \frac{1}{4\pi} \oint I d\omega \quad (6.48)$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\phi \left(\int_0^{\pi/2} I_+ \sin \theta d\omega + \int_{\pi/2}^\pi I_- \sin \theta d\omega \right) \quad (6.49)$$

$$= \frac{1}{2} (I_+ + I_-) \quad (6.50)$$

The flux is

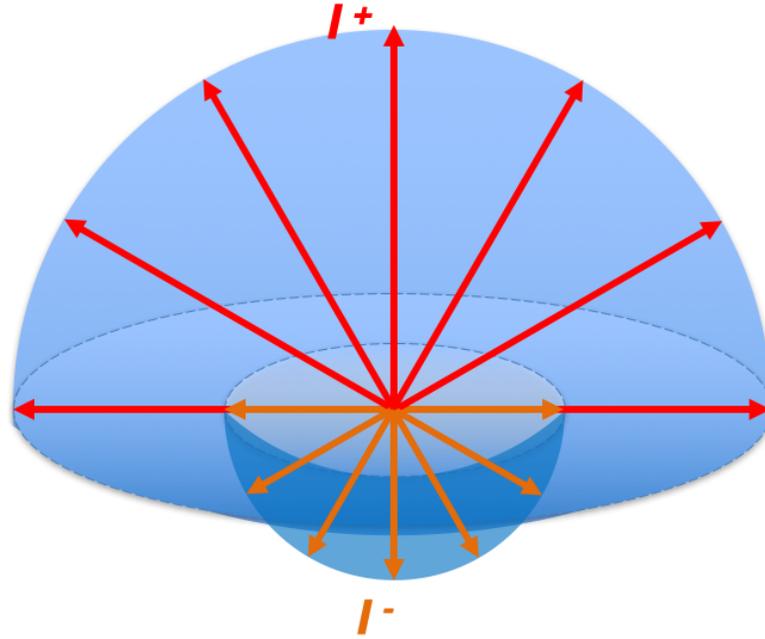


Figure 6.2: Eddington approximation.

$$F = \oint I \cos \theta d\omega \quad (6.51)$$

$$= 2\pi \left(\int_0^{\pi/2} I_+ \cos \theta \sin \theta d\omega + \int_{\pi/2}^{\pi} I_- \cos \theta \sin \theta d\omega \right) \quad (6.52)$$

$$= \pi (I_+ - I_-) \quad (6.53)$$

And the pressure is

$$P = \frac{1}{c} \oint I \cos^2 \theta d\omega \quad (6.54)$$

$$= \frac{2\pi}{c} \left(\int_0^{\pi/2} I_+ \cos^2 \theta \sin \theta d\omega + \int_{\pi/2}^{\pi} I_- \cos^2 \theta \sin \theta d\omega \right) \quad (6.55)$$

$$= \frac{4\pi}{6c} (I_+ + I_-) \quad (6.56)$$

$$(6.57)$$

We can substitute Eq. (6.50) into Eq. (6.57), uncovering a connection between pressure and mean intensity

$$P = \frac{4\pi J}{3c} \quad (6.58)$$

Now, according to the radiative equilibrium condition relating the pressure gradient and the flux (Eq. ??)

$$\frac{4\pi}{3c} J = \frac{F\tau}{c} + C \quad (6.59)$$

We can evaluate the constant of integration C by the boundary condition: at the surface $\tau = 0$ and $L = 0$. So, at the upper layer $J_{\tau=0} = F/2\pi$, and thus $C = 2F/3c$. So,

$$\frac{4\pi}{3} J = F \left(\tau + \frac{2}{3} \right) \quad (6.60)$$

We can replace the bolometric flux by Stefan-Boltzmann law $F = \sigma T_{\text{eff}}^4$, finding

$$J = \frac{3\sigma}{4\pi} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right) \quad (6.61)$$

Eq. (??) states that $S = J$, so

$$S = \frac{3\sigma}{4\pi} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right) \quad (6.62)$$

In LTE, $S_\nu = B_\nu$. We integrate this in frequency to find

$$S = \int_0^\infty B_\nu d\nu = \frac{\sigma T^4}{\pi} \quad (6.63)$$

Finally, substituting Eq. (6.63) into Eq. (6.62), we find the temperature variation in the atmosphere

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right). \quad (6.64)$$

The graphical result is shown in Fig. 6.3. This result shows that the temperature is equal to the effective temperature not at the surface ($\tau = 0$), but at the depth where $\tau = 2/3$. This stems from the fact that in the outer atmosphere of a star, the mean free path of a photon is comparable to the length scale of the temperature stratification, so we see not only a single temperature, but layers in a range of temperatures. The depth of $\tau = 2/3$ is the average point of origin of the observed photons. Though it follows from an approximation, the result applies reasonable well to real stars.

6.1.6 Stellar Spectra

Now that we understand how the temperature increases as we go deeper in a stellar atmosphere, let us understand the structure in their spectra. When stellar spectra were first obtained and the myriad of spectral lines were seen, a classification system was devised. Edward Pickering and Williamina Fleming ranked the stars alphabetically according to how dark the strong Balmer lines of hydrogen were. Stars with deepest hydrogen lines were said to be of spectral type A. The second deepest were B and so on. Any physical significance was unclear.

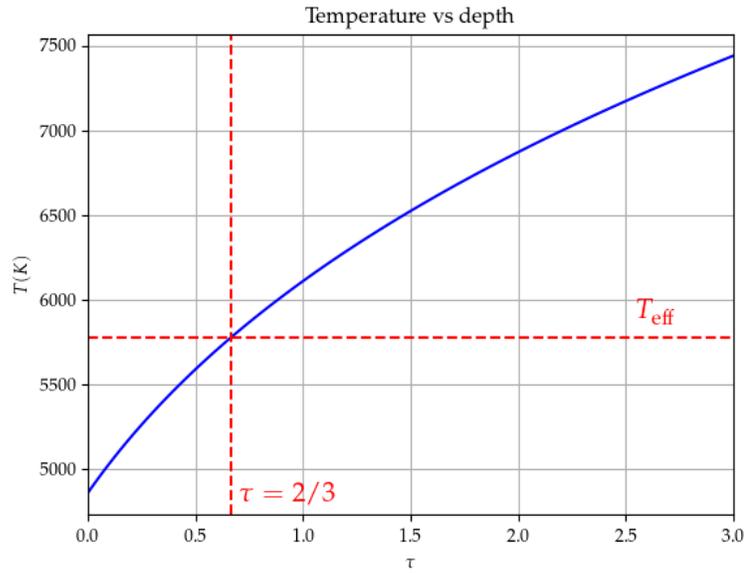


Figure 6.3: Temperature solution in the Eddington approximation.

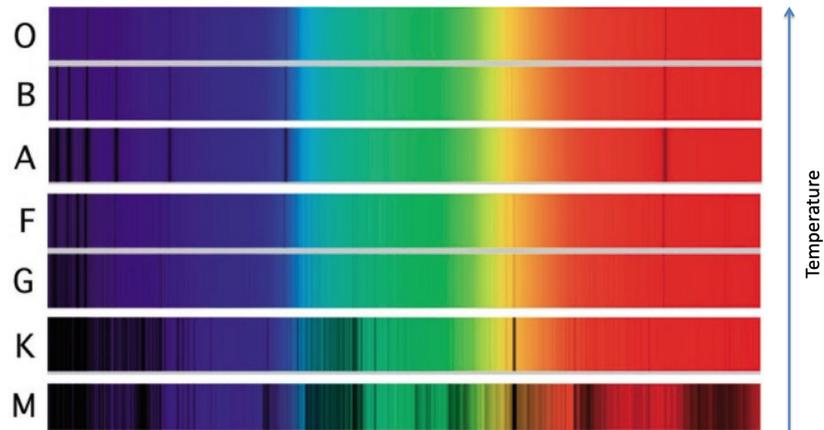


Figure 6.4: Spectral sequence. The red line is $H\alpha$, at 6563\AA . The yellow line is Na. The band of lines in the red and green in the M type are molecular lines of TiO. The strong blue lines in the A type are other hydrogen Balmer lines. The strong blue lines in F and G types are lines of Ca II.

As it turns out, the sequence is actually a temperature sequence, with O and B stars out of place because increasing ionization with increasing temperature starts to weaken the hydrogen lines. Another astronomer, Annie Jump Cannon, simplified the system and grouped them as a series of decreasing temperature as OBAFGKM Fig. 6.4. The sequence can be derived from local thermal equilibrium combining quantum level populations and statistical mechanics. The full description is known as it Saha equation.

6.1.6.1 Saha Equation

The Balmer lines peak at the spectral type A, for which the temperature of the atmosphere is around 10 000 K. As the Balmer lines are transitions from the $n = 2$ state, a significant number of atoms should be required at that state.

The populations of atoms in a given excited states is given by Boltzmann excitation equation, Eq. (5.133). Given states a and b , their ratio is

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} \exp\left[-\frac{(E_b - E_a)}{kT}\right] \quad (6.65)$$

Given that the hydrogen energy levels are $E_n = -13.6\text{eV}/n^2$, and the level degeneracy is $g_n = 2n^2$, we can calculate the ratio of atoms at the $N = 2$ level relative to the $N = 1$ level as a function of temperature.

$$\frac{N_2}{N_1} = 4 \exp\left[-\frac{118604 \text{ K}}{T}\right] \quad (6.66)$$

where the coefficient 4 is the ratio of the degeneracies and the temperature nearing 10^5 K is $313.6\text{eV}/4k$, remembering that the Boltzmann constant k in units of electron-volt per kelvin is $k \approx 8.6 \times 10^{-5} \text{ eV/K}$.

The result for an idealized hydrogen atom of only two levels is shown in Fig. 6.5. The crossover, $N_2 = N_1$, occurs at about 10^5 K, as expected from the argument of the exponential Eq. (6.66). The conclusion is that very high temperatures are necessary for a significant number of atoms to be in the first excited state. At 10^4 K, less than a thousand of the atoms are at the N_2 level, and according to Eq. (6.66), the line strength should increase with the temperature. Why, then, the Balmer lines peak at 10^4 K?

The problem, as may have been guessed already, is that at high temperatures, the atoms will start to ionize. At too low temperatures few atoms will be at the excited state, but at too high temperatures, also few atom will be at the excited stated because ionization is too high.

We can write Boltzmann levels for the ionization balance. The form is the same as Eq. (6.65), with n_b and n_a now being two ionization states of the same element. The ionization potential, i.e. the energy needed to ionize i from the ground state is χ , and the degeneracy of the ground states of the two ions are g_i and g_{i+1} , respectively.

The degeneracy of an ion in the lower ionization state in Eq. (6.65) is just g_i . As for the upper ionization state the situation is trickier as we now have an ion and a free electron; the degeneracy of the ionized state will be $g_{i+1}g_e$, i.e., the degeneracy of the ion times the degeneracy of the free electron, i.e., the number of possible states in which a free electron may be put. For hydrogen, g_{i+1} is simply 1: since there is no electron left, there is only a single state. If we assume i to be the ground state of neutral hydrogen, the statistical weight is 2, so $g_{i+1}/g_i = 1/2$. The last piece missing to find the ionization balance is g_e , the degeneracy of the free electron.

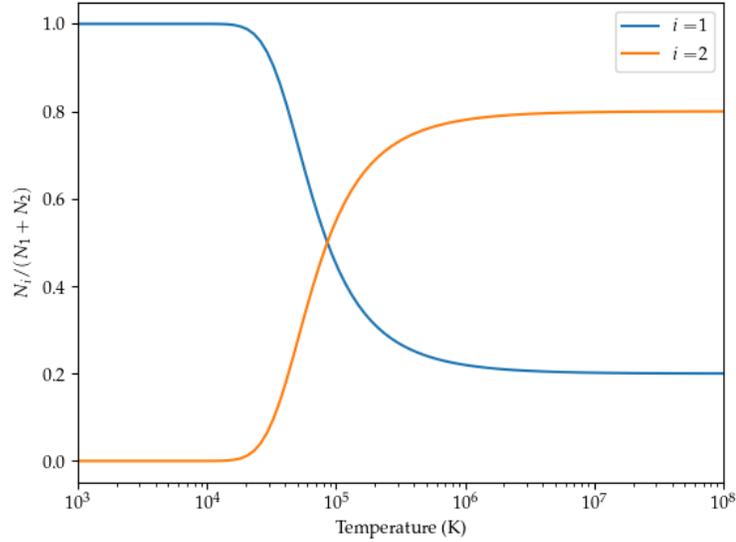


Figure 6.5:

6.1.6.2 Degeneracy of a free electron

Let us calculate the number of states that the free electron can occupy. Because of the uncertainty principle, the smallest combination of space and momentum of the free electron must obey

$$dx dp = h \quad (6.67)$$

which means that the smallest volume the electron can occupy is

$$d^3x = \frac{h^3}{d^3p} \quad (6.68)$$

Let us call this the smallest “cell” of space that an electron with a given momentum between p and $p + dp$ can occupy. The number dN of such cells available is the volume V_e occupied by the electron distribution, in units of this elementary cell, i.e.,

$$dN = \frac{V_e}{dx^3} = \frac{V}{h^3} d^3p \quad (6.69)$$

Because of Pauli’s exclusion principle, each of these cells can hold two electrons, one of spin up and one of spin down. The number dg_e of electron states is $dg_e = 2dN$. This number is infinitesimal because it captures only electrons with momentum between p and $p + dp$.

Assuming spherical symmetry $d^3p = 4\pi p^2 dp$, and the volume of the electron distribution is the inverse of the number density $V_e = 1/n_e^3$. Substituting these expressions,

$$dg_e = \frac{8\pi p^2}{n_e h^3} dp \quad (6.70)$$

Because this number is infinitesimal it gives us only dN_{i+1} , not the full number of ions in the upper level.

$$\frac{dN_{i+1}}{N_i} = \frac{g_{i+1}}{g_i} dg_e \exp\left[-\frac{(E_{i+1} - E_i)}{kT}\right] \quad (6.71)$$

In the ionized state the electron is unbounded, thus the energy E_{i+1} is simply the kinetic energy $p^2/2m$ of the free electron. In the bound state, the electron had energy $-\chi$ where χ is the ionization potential. Thus, $E_{i+1} - E_i = p^2/2m + \chi$ and Eq. (6.71) can be written

$$\frac{dN_{i+1}}{N_i} = \frac{g_{i+1}}{g_i} \frac{8\pi}{n_e h^3} \exp\left[-\frac{(p^2/2m + \chi)}{kT}\right] p^2 dp \quad (6.72)$$

This expression still only gives the infinitesimal number of states where the free electron has momentum between p and $p + dp$. We need to integrate this in momentum to find the total number of states

$$\frac{N_{i+1}}{N_i} = \frac{1}{N_i} \int dN_{i+1} \quad (6.73)$$

$$= \frac{g_{i+1}}{g_i} \frac{8\pi}{n_e h^3} e^{-\chi/kT} \int_0^\infty \exp[-p^2/2mkT] p^2 dp \quad (6.74)$$

The integral is

$$\int_0^\infty \exp[-p^2/2mkT] p^2 dp = \frac{\sqrt{\pi}}{4} (2mkT)^{3/2} \quad (6.75)$$

So Eq. (6.76) becomes

$$\frac{N_{i+1}}{N_i} = \frac{g_{i+1}}{g_i} \frac{2}{n_e h^3} e^{-\chi/kT} (2\pi mkT)^{3/2} \quad (6.76)$$

We can express the ratio in the LHS in terms of number densities, to finally find the Saha ionization equation

$$\boxed{\frac{n_{i+1} n_e}{n_i} = \frac{2g_{i+1}}{g_i} \left[\frac{(2\pi mkT)^{1/2}}{h} \right]^3 e^{-\chi/kT}} \quad (6.77)$$

For hydrogen, the states are $n_i = n_{\text{HI}}$ (the neutral state), $n_{i+1} = n_{\text{HII}}$ (ionized state), and $n_e = n_{\text{HII}}$ because hydrogen has only one electron. The degeneracies are $2g_{i+1}/g_i = 1$, so the Saha equation reduces to

$$\frac{n_{\text{HII}}^2}{n_{\text{HI}}} = \left[\frac{(2\pi mkT)^{1/2}}{h} \right]^3 e^{-\chi/kT} \quad (6.78)$$

The ionization fraction $x = n_{\text{HII}}/(n_{\text{HI}} + n_{\text{HII}})$ is obtained by writing

$$\frac{n_{\text{HII}}^2}{n_{\text{HI}}} = \frac{n_{\text{HII}}^2}{n_{\text{H}} - n_{\text{HII}}} = n_{\text{H}} \frac{(n_{\text{HII}}/n_{\text{H}})^2}{(1 - n_{\text{HII}}/n_{\text{H}})} = n_{\text{H}} \frac{x^2}{1 - x} \quad (6.79)$$

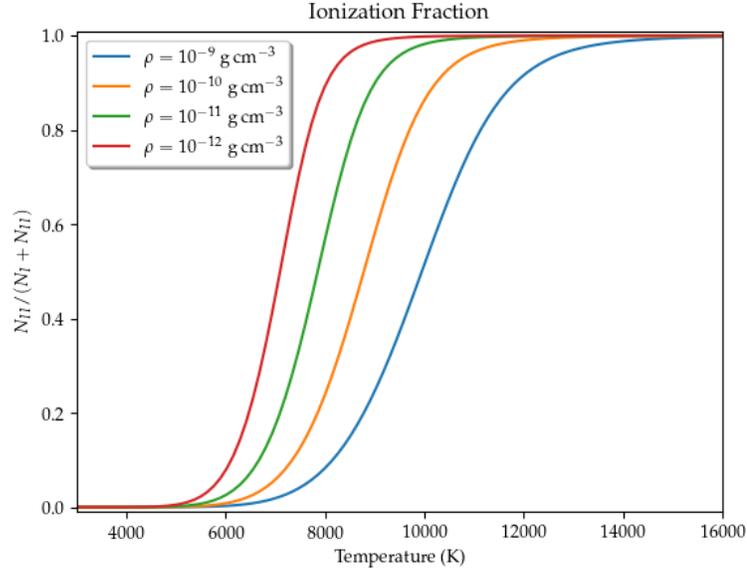


Figure 6.6:

where $n_H = n_{\text{HI}} + n_{\text{HII}}$ is the number density of the gas. The number density in terms of the mass density is $n_H = \rho/m_H$, so we finally write

$$\frac{n_{\text{HII}}^2}{n_{\text{HI}}} = \frac{\rho}{m_H} \frac{x^2}{1-x}. \quad (6.80)$$

This equation is a quadratic

$$x^2 + \alpha x - \alpha = 0 \quad (6.81)$$

with

$$\alpha \equiv \frac{m_H n_{\text{HII}}^2}{\rho n_{\text{HI}}}. \quad (6.82)$$

The solution is thus

$$x = \frac{\sqrt{\alpha^2 + 4\alpha} - \alpha}{2}, \quad (6.83)$$

where we discarded the unphysical negative root. The solution is shown in Fig. 6.6 for a set of densities, and using $\chi = 13.6$ eV for the ionization potential of hydrogen.

The strength of the Balmer lines depends on the population of atoms in the first excited state, relative to the total number of hydrogen atoms. Combining Boltzmann levels with Saha equation

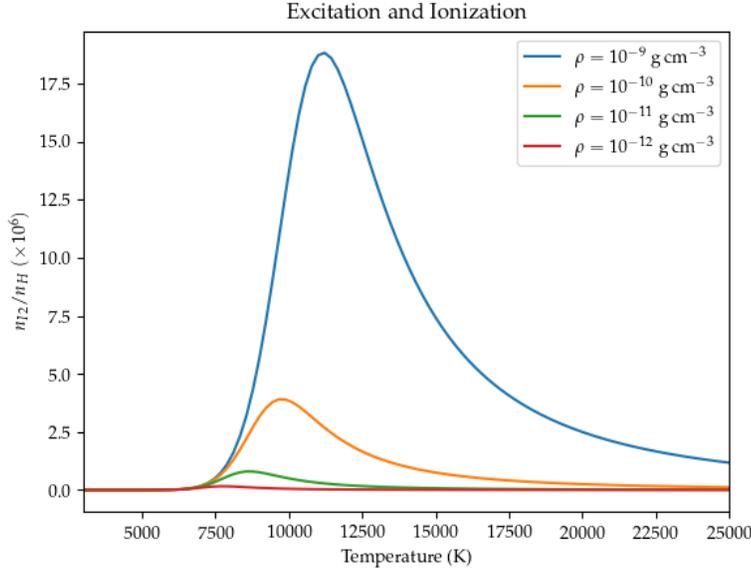


Figure 6.7:

$$\frac{n_{I2}}{n_H} = \frac{n_{I2}}{n_I} \frac{n_I}{n_H} \quad (6.84)$$

The result is shown in Fig. 6.7. The population of atoms at the N=2 level, and thus the Balmer lines, peak at about 10^4 K, the inflection point of the ionization profile.

6.2 Stellar Structure

We will now derive the equations that determine the interior structure of a star. Consider the amount of mass in an infinitesimal volume element

$$\begin{aligned} dm &= \rho dV \\ &= \rho r^2 \sin \theta dr d\theta d\phi \end{aligned} \quad (6.85)$$

If we consider spherical symmetry, we can integrate in solid angle, leaving only the dependency in radius. The mass in a shell of radius r and thickness dr is

$$dM(r) = \oint_{\omega} dm(r) = \rho(r)r^2 dr \oint \sin \theta d\theta d\phi = 4\pi r^2 \rho(r) dr \quad (6.86)$$

Dividing by dr , we have the *mass continuity equation*

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho} \quad (6.87)$$

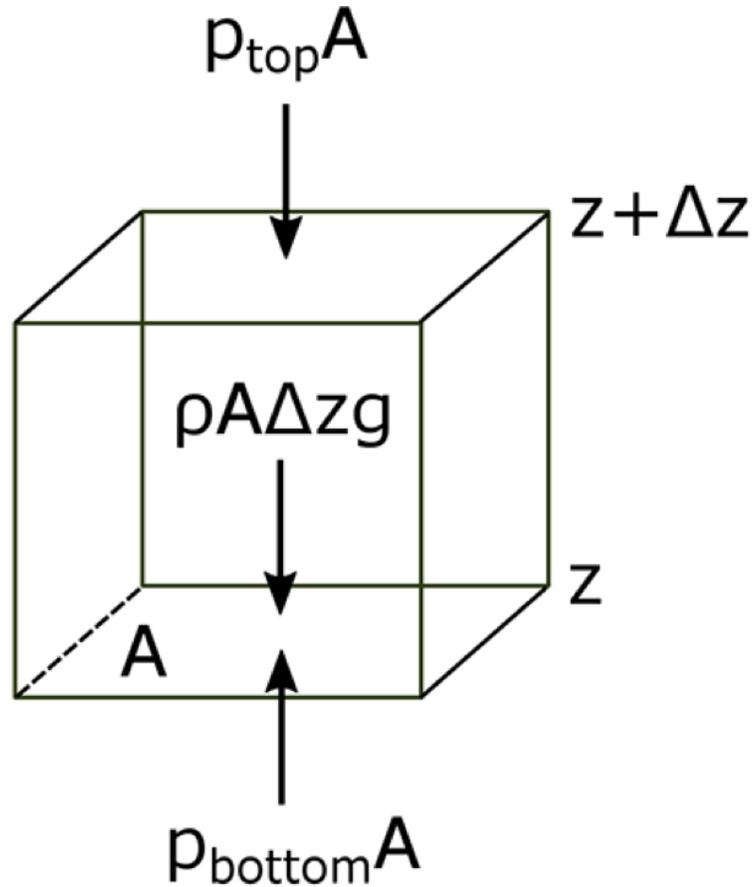


Figure 6.8:

We can also derive an equation for the force balance. A gas parcel feels a gravity force towards the center of star, given by the mass inside its shell

$$dF_g = -\frac{GM_r dm}{r^2} \quad (6.88)$$

where $M_r = \int_0^r 4\pi\rho r'^2 dr'$ and m is the mass of the gas parcel. This gravity has to be balanced by the pressure force, the difference in pressure from the base of the gas parcel to its top (Fig. 6.8). Considering the gas parcel to have area dA , the force above is $F_{\text{top}} = PdA$ and the force below is $F_{\text{bottom}} = (P + dP)dA$. The difference is thus

$$dF_p = -dPdA \quad (6.89)$$

We now consider the forces to be in balance

$$dPdA = -\frac{GM_r dm}{r^2} \quad (6.90)$$

and considering that $dm = \rho dA dr$, we find the *equation of hydrostatic equilibrium*

$$\boxed{\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho} \quad (6.91)$$

6.2.1 A constant density star

Let us solve these equations for the idealized case of a star of constant density. The case is artificial, but some order of magnitude insight will be obtained.

If the density is constant, we can replace $\rho = 3M/4\pi R^3$, where M and R are the mass and radius of the star, respectively. The mass in a shell, then is

$$M_r = \frac{4\pi}{3} r^3 \rho = \frac{r^3}{R^3} M \quad (6.92)$$

We can use this result to substitute M_r in the pressure equation

$$dP = -\frac{GM\rho r}{R^3} dr. \quad (6.93)$$

We can also use the mass continuity equation to replace dr by dM_r , using $dM_r = 4\pi r^2 \rho dr$. This leads to

$$dP = -\frac{GM}{4\pi R^3 r} dM_r \quad (6.94)$$

and we can use Eq. (6.92) to substitute the radius

$$dP = -\frac{GM^{4/3}}{4\pi R^4} M_r^{-1/3} dM_r \quad (6.95)$$

We can now integrate to find the pressure. We need to define the integration interval. We can integrate from the surface to an arbitrary location r . At the arbitrary location the pressure is P_r and the mass is M_r . At the surface the pressure is zero, and the mass is M .

$$\int_0^{P_r} dP = -\frac{GM^{4/3}}{4\pi R^4} \int_M^{M_r} M_r'^{-1/3} dM_r' \quad (6.96)$$

$$P_r = -\frac{3GM^{4/3}}{8\pi R^4} (M_r^{2/3} - M^{2/3}) \quad (6.97)$$

That is

$$P_r = \frac{3GM^2}{8\pi R^4} \left[1 - \left(\frac{M_r}{M} \right)^{2/3} \right] \quad (6.98)$$

$$= P_c \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad (6.99)$$

Where P_c is the central pressure, at $r = 0$

$$P_c = \frac{3GM^2}{8\pi R^4} \quad (6.100)$$

Substituting the constants, the central pressure can be expressed as

$$P_c = 1.34 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{ dyne cm}^{-2} \quad (6.101)$$

$$\approx 10^9 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{ atm} \quad (6.102)$$

Substituting the equation of state, we have the temperature

$$P = c_v \rho T \quad (6.103)$$

$$T_c = \frac{1}{2} \frac{GM}{R} \frac{\mu}{N_A k} \approx 10^7 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} K \quad (6.104)$$

Which should be the order of magnitude of the temperature of the solar core. Back in chapter 5 we highlighted that the presence of absorption lines in the solar spectrum shows that the Sun is a case of Kirchhoff's 3rd law, where the atmosphere is a colder gas superimposed on a hotter lamp. Eq. (6.104) shows just how much hotter this lamp is.

6.2.2 Luminosity equation

We can derive an equation for the luminosity also in terms of elementary considerations. Define the energy production rate ε as energy per unit time per mass. Its unit is

$$[\varepsilon] = \text{erg s}^{-1} \text{ g}^{-1} \quad (6.105)$$

In a spherical shell the luminosity is thus

$$dL_r = 4\pi r^2 dr \rho \varepsilon \quad (6.106)$$

dividing by dr , we find the luminosity equation

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon}. \quad (6.107)$$

6.2.3 Stellar energy

We have derived the equations of mass continuity and pressure equilibrium. We also found an equation for the luminosity, that depends on the energy production rate ε . Yet, we do not know anything yet about how this energy is generated. One possibility, that was seriously entertained in the 19th century was the *release of gravitational binding energy*. We know this hypothesis to be flawed, and that stars derive their energy from nuclear fusion happening at their cores. Yet, although it does not power stars, the release of gravitational binding energy is an actual source of energy in many other astrophysical objects. Therefore, it is warranted to introduce the concept.

A star is a gravitationally bound object. Its total gravitational energy is defined as the negative of the energy required to separate its constituent particles to infinity. Let us calculate this energy. Suppose we have a spherical shell of mass dM . To move this shell from radius r to $r+dr$ requires $F \cdot dr$ units of work. The force F is the gravitational force, so the energy dW needed to move it from r to infinity is

$$dW = \int_r^\infty \frac{GM_r}{r'^2} dM dr' \quad (6.108)$$

$$= -\frac{GM_r}{r} dM_r \quad (6.109)$$

To disperse the whole star requires that we do that for all dM , or

$$W = \int dW = - \int_0^M \frac{GM_r}{r} dM_r \quad (6.110)$$

We can write that as $W = -qGM^2/R$, where q is of order unity and reflects the mass distribution within the star. For constant density,

$$dM_r = 4\pi r^2 \rho dr \quad (6.111)$$

$$M_r = \frac{r^3}{R^3} M \quad (6.112)$$

Then

$$W = -\frac{4\pi G \rho M}{R^3} \int_0^R r^4 dr \quad (6.113)$$

$$= -\frac{4\pi}{5} G \rho M R^2 \quad (6.114)$$

And because $\rho = 3M/4\pi R^3$, we have

$$W = -\frac{3}{5} \frac{GM^2}{R} \quad (6.115)$$

i.e., $q = 3/5$ for constant density.

6.2.3.1 Virial Theorem

Consider the scalar $\sum_i p_i r_i$. Derivate it in time so that

$$\frac{d}{dt} \sum_i p_i r_i = \sum_i \dot{p}_i r_i + p_i \dot{r}_i \quad (6.116)$$

The last term is $\sum_i m_i v_i^2$, equal to twice the total kinetic energy of the star, $2K$. Recalling also that $\dot{p}_i = F_i$, the first term is $\sum_i \dot{F}_i r_i$, or the work done by the gravitational force. This is the quantity W derived first. So,

$$2K + W = \frac{d}{dt} \sum_i p_i r_i \quad (6.117)$$

We can evaluate the time derivative by noticing this is

$$\frac{d}{dt} \sum_i m_i \dot{r}_i r_i = \frac{1}{2} \frac{d}{dt} \sum_i \frac{d}{dt} (m_i r_i^2) = \frac{1}{2} \frac{d^2 I}{dt^2} \quad (6.118)$$

where I is the inertia moment. Substituting Eq. (6.118) into Eq. (6.117),

$$2K + W = \frac{1}{2} \frac{d^2 I}{dt^2} \quad (6.119)$$

a result known as *the virial theorem*. If the system is in steady state, or if the contraction is sufficiently small, we can set the RHS to zero, and write the more familiar form of the virial theorem

$$2K + W = 0 \quad (6.120)$$

According to this form of the virial theorem, the total energy is half the gravitational potential energy

$$E = K + W = \frac{W}{2} = -\frac{|W|}{2} \quad (6.121)$$

Notice also that as a star steadily contracts, it becomes more gravitationally bound, making W larger, and it follows that K also becomes larger, so the star becomes hotter. By how much? Suppose the contracting star increases its boundness by dW . According to Eq. (6.120), the change in temperature is

$$2(K + \Delta K) + (W + \Delta W) = 0 \quad (6.122)$$

$$\Delta K = -\frac{1}{2} \Delta W = \frac{1}{2} |\Delta W| \quad (6.123)$$

As for the total energy,

$$E = (K + \Delta K) + (W + \Delta W) \quad (6.124)$$

$$= E_0 + (\Delta K + \Delta W) \quad (6.125)$$

$$= E_0 + \left(-\frac{1}{2} \Delta W + \Delta W\right) \quad (6.126)$$

$$= E_0 + \frac{1}{2} \Delta W \quad (6.127)$$

$$= E_0 - \frac{1}{2} |\Delta W| \quad (6.128)$$

The energy of the star was decreased by $\Delta E = -|\Delta W|/2$. This is a puzzling result. Energy is conserved, yet we find that the star decreased its energy upon contraction. Where did this energy go?

It must have left the system as *radiation*.

This leads to a very elegant result: as a star contracts, according to Eq. (6.123), half of the energy removed from the gravitational field remains in the system as kinetic energy (increasing the temperature) and according to Eq. (6.128) half is radiated away. *Gravitational contraction leads to release of radiation*. Helmholtz and Kelvin suggested that this is how stars shine. For the Sun,

$$\frac{GM_{\odot}^2}{R_{\odot}} \approx 3.8 \times 10^{48} \text{ erg.} \quad (6.129)$$

Dividing it by the solar luminosity, we get a timescale of about 10^7 yrs, 10 million years, which cannot be the whole story (Kelvin thought that it was, and never accepted that the Earth could be older than that, in spite of geological and fossil evidence).

6.2.4 Temperature equation

Knowing the luminosity

$$L_r = 4\pi r^2 F_r \quad (6.130)$$

the condition of radiative equilibrium gives an equation for the temperature. In here, F_r is given by the condition of radiative equilibrium $\nabla \cdot \mathbf{F} = \partial_r F = 0$, which yields

$$F_r = -\frac{16}{3} \frac{\sigma T^3}{\kappa_R \rho} \frac{dT}{dr} \quad (6.131)$$

So we can write

$$\frac{dT}{dr} = -\frac{3\kappa_R \rho L_r}{16\pi r^2 c a_R T^3} \quad (6.132)$$

where we substituted $a_R = \sigma/4c$. This is the temperature equation if the energy is transported by radiation.

6.2.4.1 Convection

Let us consider a slab of the interior of the star, hotter below and colder above, and in this slab, the upward displacement of a gas bubble (Fig. 6.9).

We will use prime superscripts for the cold slab, plain for the hot, and star superscript for the bubble, as shown in Fig. 6.9. The bubble expands adiabatically

$$\rho^* = \rho \left(\frac{P^*}{P} \right)^{1/\gamma} \quad (6.133)$$

The surrounding pressure and density are

$$P' = P + \frac{dP}{dr} \Delta r \quad (6.134)$$

$$\rho' = \rho + \frac{d\rho}{dr} \Delta r \quad (6.135)$$

In addition, the bubble is always in pressure equilibrium

$$P^* = P' \quad (6.136)$$

So

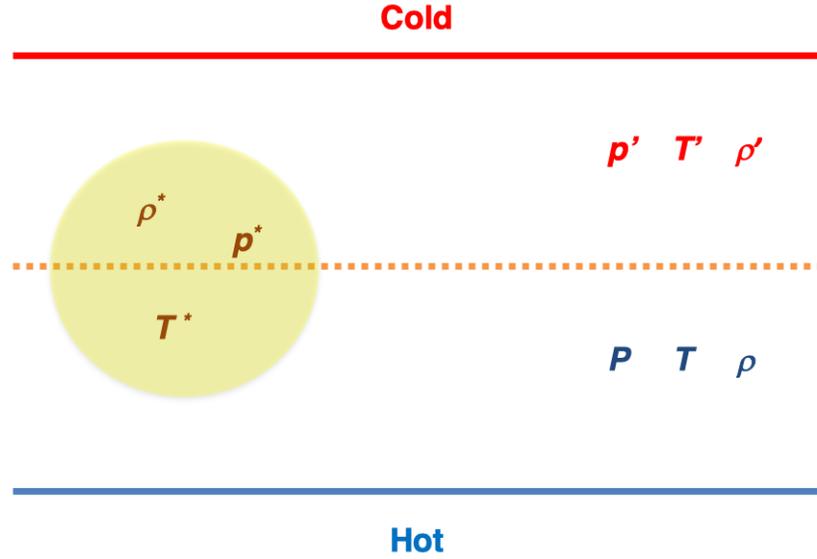


Figure 6.9: Convection.

$$\rho^* = \rho \left(\frac{P'}{P} \right)^{1/\gamma} \quad (6.137)$$

$$= \rho \left(1 + \frac{1}{P} \frac{dP}{dr} \Delta r \right)^{1/\gamma} \quad (6.138)$$

$$= \rho \left(1 + \frac{1}{\gamma P} \frac{dP}{dr} \Delta r \right) \quad (6.139)$$

$$(6.140)$$

Recall now the equation of state, $P = \rho RT$, and write it in differential form

$$d \ln \rho = d \ln P - d \ln T \quad (6.141)$$

Divide this by dr

$$\frac{d\rho}{\rho} = \frac{\rho}{P} \frac{dP}{dr} - \frac{\rho}{T} \frac{dT}{dr} \quad (6.142)$$

And substitute in the equation for the surrounding density ρ'

$$\rho' = \rho + \left(\frac{\rho}{P} \frac{dP}{dr} - \frac{\rho}{T} \frac{dT}{dr} \right) \Delta r \quad (6.143)$$

Convection will set in if the bubble keeps rising. That is, if $\rho^* < \rho'$. The surrounding pressure is

$$\rho' = \rho + \left(\frac{\rho}{P} \frac{dP}{dr} - \frac{\rho}{T} \frac{dT}{dr} \right) \Delta r \quad (6.144)$$

The bubble pressure is

$$\rho^* = \rho + \frac{\rho}{\gamma P} \frac{dP}{dr} \Delta r \quad (6.145)$$

Therefore,

$$\rho^* - \rho' = \frac{\rho}{\gamma P} \frac{dP}{dr} \Delta r - \left(\frac{\rho}{P} \frac{dP}{dr} - \frac{\rho}{T} \frac{dT}{dr} \right) \Delta r \quad (6.146)$$

And the condition for buoyancy $\delta\rho = \rho^* - \rho' < 0$, is

$$\frac{dT}{dr} < \frac{T}{P} \left(1 - \frac{1}{\gamma} \right) \frac{dP}{dr} \quad (6.147)$$

which is the *Schwarzschild instability condition*.

As these gradients are negative, we can write it as

$$\left| \frac{dT}{dr} \right| > \left(1 - \frac{1}{\gamma} \right) \frac{T}{P} \left| \frac{dP}{dr} \right| \quad (6.148)$$

This defines the adiabatic temperature gradient, of marginal stability

$$\left. \frac{dT}{dr} \right|_{\text{ad}} \equiv \left(1 - \frac{1}{\gamma} \right) \frac{T}{P} \frac{dP}{dr} \quad (6.149)$$

And the condition for convection is that the gradient is steeper than adiabatic

$$\left| \frac{dT}{dr} \right| > \left. \frac{dT}{dr} \right|_{\text{ad}} \quad (6.150)$$

We can also write

$$\left. \frac{d \ln T}{d \ln r} \right|_{\text{ad}} = \left(1 - \frac{1}{\gamma} \right) \left. \frac{d \ln P}{d \ln r} \right| \quad (6.151)$$

A simpler way to write this is

$$\left(\frac{d \ln T}{d \ln P} \right)_{\text{ad}} = 1 - \frac{1}{\gamma} \quad (6.152)$$

which removes the need to keep track of signs because dT and dP have the same sign, i.e., T and P increase in the same direction. Some authors like to call the quantity in the LHS ∇_{ad} , so

$$\nabla_{\text{ad}} \equiv \left(\frac{d \ln T}{d \ln P} \right)_{\text{ad}} = 1 - \frac{1}{\gamma} \quad (6.153)$$

We can also write in general

$$\nabla \equiv \left(\frac{d \ln T}{d \ln P} \right) \quad (6.154)$$

and the condition for convection becomes

$$\nabla > \nabla_{\text{ad}} \quad (6.155)$$

The notation, although confusing, is compact and widespread. The radiative gradient is

$$\nabla_{\text{rad}} \equiv \left(\frac{d \ln T}{d \ln P} \right)_{\text{rad}} \equiv \frac{3P_{\kappa} L_r}{16\pi acGT^4 M_r} \quad (6.156)$$

and with that we have the full set of equations of stellar structure

$$\frac{dM}{dr} = 4\pi r^2 \rho \quad (6.157)$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho \quad (6.158)$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon \quad (6.159)$$

$$\frac{dT_r}{dr} = \begin{cases} -\frac{3\kappa_R \rho L_r}{16\pi r^2 c a R T^3} & \text{for } \nabla < \nabla_{\text{ad}} \\ \frac{T}{P} \left(1 - \frac{1}{\gamma}\right) \frac{dP}{dr} & \text{for } \nabla \geq \nabla_{\text{ad}} \end{cases} \quad (6.160)$$

6.2.5 The Lane-Emden equation and polytropes

We found a set of equations for several fundamental variables in stellar interiors. Yet, they all depend on the density, and we do not yet have an equation for the density. That is because the density follows the equation of state, relating pressure, density, and temperature, $p = p(\rho, T)$, so when two of these quantities are specified, the third one is too. Yet, there is a useful equation that allows us to identify some properties of stellar interiors, the Lane-Emden equation. Taking the derivative of the equation of hydrostatic equilibrium, we find

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -G \frac{dM_r}{dr} \quad (6.161)$$

and substituting the equation of mass continuity

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho \quad (6.162)$$

The RHS is the same right hand side of the Poisson equation for the gravitational potential, $\nabla^2 \Phi = 4\pi G \rho$, which in spherical coordinates is

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho \quad (6.163)$$

Comparing Eq. (6.162) and Eq. (6.163), we see that

$$\frac{1}{\rho} \frac{dP}{dr} = \frac{d\Phi}{dr} \quad (6.164)$$

as expected from a situation of hydrostatic balance. If we know use the following ansatz for the equation of state

$$P = K\rho^\gamma \quad (6.165)$$

then Eq. (6.162) becomes

$$\frac{\gamma K}{r^2} \frac{d}{dr} \left(r^2 \rho^{\gamma-2} \frac{d\rho}{dr} \right) = -4\pi G \rho \quad (6.166)$$

If $\gamma = 2$ then the LHS would be a perfect Laplacian. This is not always the case, but we can use a mathematical trick to zero the exponent on the density inside the first derivative. We can replace

$$\rho = \rho_c \theta(r)^n \quad (6.167)$$

where ρ_c is a constant (the central density), and $\theta(r)$ is a dimensionless function that carries the radial dependency of the density. The density derivative becomes

$$\frac{d\rho}{dr} = \rho_c n \theta^{n-1} \frac{d\theta}{dr} \quad (6.168)$$

and

$$\frac{\gamma n K \rho_c^{\gamma-2}}{4\pi G r^2} \frac{d}{dr} \left(r^2 \theta^{n(\gamma-2)+n-1} \frac{d\theta}{dr} \right) = -\theta^n \quad (6.169)$$

To have a Laplacian in the LHS, the need to cancel the exponent of θ , that is

$$\gamma = \frac{n+1}{n} \quad (6.170)$$

and Eq. (6.171) becomes

$$\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) = -\theta^n \quad (6.171)$$

The constants on the LHS have dimension of distance squared. So, we can define a length

$$\lambda = \left[\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G} \right]^{1/2} \quad (6.172)$$

and replace the radius r by $r = \lambda \xi$ where ξ is dimensionless. Doing so, we find the Lane-Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n. \quad (6.173)$$

Solving for θ , we have the density, and via Eq. (6.165), also P . If the ideal gas law holds, we also find the temperature. Notice that using Eq. (6.170), we can write Eq. (6.165) as

$$P = K \rho^{(n+1)/n} \quad (6.174)$$

Eq. (6.174) is known as the *polytropic* equation of state. The Sun is very well approximated by a polytrope of index $n = 3$. Low-mass stars by polytropes of index $n = 3/2$. Jupiter and Saturn are well approximated by polytropes of index $n = 1$.

The Lane-Emden equation has analytical solutions for only three integer values of n

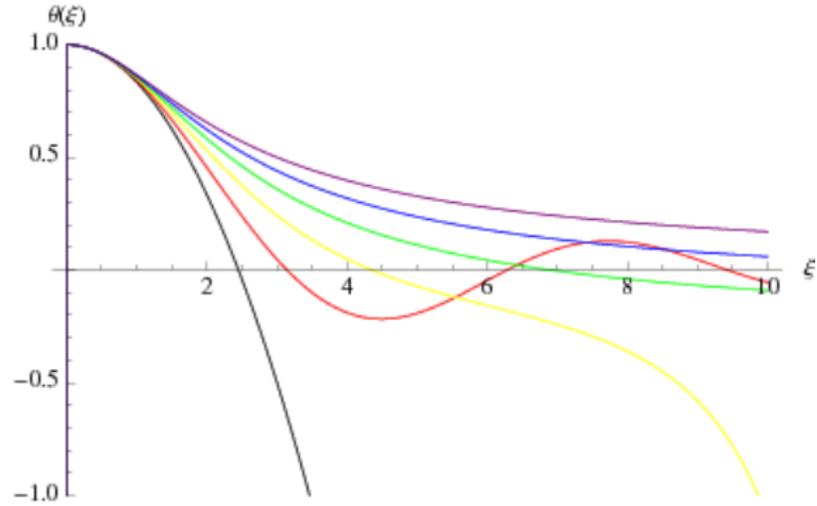


Figure 6.10: Solutions of the Lane-Endem equation for $n=0$ (black), $n=1$ (red), $n=2$ (yellow), $n=3$ (green), $n=4$ (blue), and $n=5$ (violet).

$$n = 0 \rightarrow \theta = 1 - \frac{\xi^2}{6} \quad (6.175)$$

$$n = 1 \rightarrow \theta = \frac{\sin \xi}{\xi} \quad (6.176)$$

$$n = 5 \rightarrow \theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \quad (6.177)$$

otherwise the solution has to be found numerically. Some solutions are plotted in Fig. 6.10.

Essentially, the Lane-Endem equation works as an equation of state. In this absence, the density has to be derived from the pressure and temperature via the ideal gas law as one solves the equations of stellar structure.

A full solar model is shown in Fig. 6.11, assuming that the energy is generated in the inner 20% of the Sun. Notice how the luminosity increases in this region, and then maintains a constant level. The interior of the Sun simply transports the luminosity produced in the core.

6.3 Stellar Evolution

6.3.1 Stellar Lifetimes

Consider again the equations of stellar structure. Taking the hydrostatic equilibrium equation (Eq. 6.91), let us make the crude approximation that $dP/dr \sim P/R$, i.e.

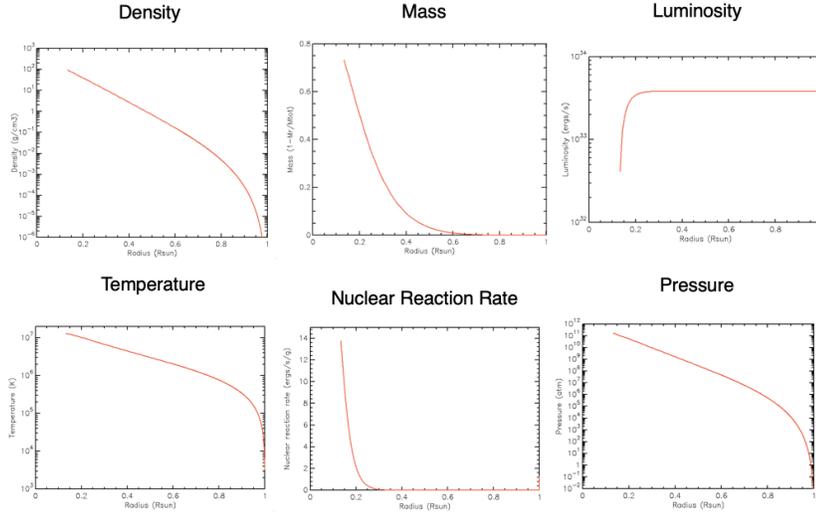


Figure 6.11: A solar model.

$$\frac{P}{R} \propto \frac{M}{R^2} \rho. \quad (6.178)$$

Then assuming constant density

$$\rho \propto \frac{M}{R^3} \quad (6.179)$$

and substituting Eq. (6.179) into Eq. (6.178) we find the following scaling for the pressure

$$P \propto \frac{M}{R} \rho \propto \frac{M^2}{R^4}. \quad (6.180)$$

Yet, the equation of state says that $P \propto \rho T$, and substituting Eq. (6.179)

$$P \propto \frac{M}{R^3} T \quad (6.181)$$

To have both Eq. (6.180) and Eq. (6.181) hold, we must have

$$\frac{M^2}{R^4} \propto \frac{MT}{R^3} \quad (6.182)$$

finally, isolating the temperature

$$T \propto \frac{M}{R}. \quad (6.183)$$

If we play the same game with the (radiative) temperature equation (Eq. 6.132), approximating $dT/dr \sim T/R$,

$$\frac{T}{R} \propto \frac{M}{R^3} \frac{L}{R^2} \quad (6.184)$$

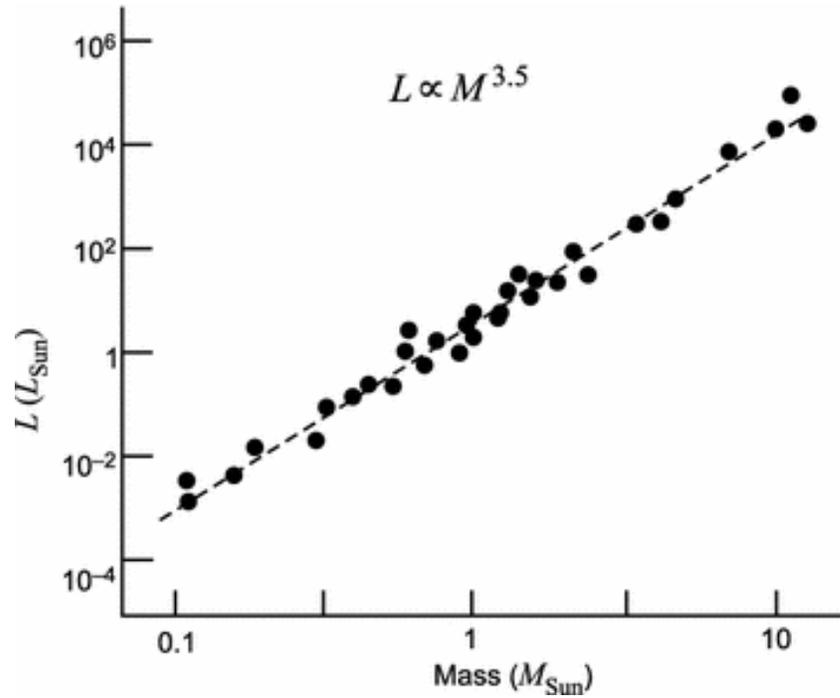


Figure 6.12: Mass luminosity relationship in the main sequence.

and isolating the luminosity

$$L \propto \frac{T^4 R^4}{M}. \quad (6.185)$$

Finally, substituting Eq. (6.183) into Eq. (6.185)

$$L \propto M^3 \quad (6.186)$$

i.e., the stellar luminosity increases with the cube of the stellar mass. The actual mass-luminosity relation for real stars is shown in Fig. 6.12. For main sequence stars, the mass-luminosity relation is closer to an exponent of 3.5. An impressive agreement considering our crude approximations.

From Eq. (6.186) we can derive the lifetime of a star. If we assume that the lifetime is the energy over the luminosity (i.e., the amount of fuel divided by the rate the fuel is consumed), and that the energy is proportional to the mass

$$t_{\text{life}} \equiv \frac{E}{\dot{E}} \sim \frac{M}{L} \quad (6.187)$$

Then, substituting Eq. (6.186),

$$\boxed{t_{\text{life}} \propto M^{-2}} \quad (6.188)$$

This is a fantastic result, and even the more remarkable considering the simple approximations done. The lifetime of a star decreases with the square of its mass. A more massive star has more fuel, but it burns it at a faster rate.

We turn now to the question of what exactly is this “fuel”.

6.3.2 Nucleosynthesis

A nucleus is always found to be less massive than the combined individual masses of protons and neutrons (collective called nucleons). Thinking in terms of energy-mass equivalence, this observation means that for the nucleons, being bound in as a nucleus represents a lower energy state than being free. So, we can expect that free nucleons when fusing in a bound nucleus is a process similar to a system jumping to a lower quantum state, with the energy difference getting radiated away.

How much energy is this? Define the atomic mass as A , and the atomic number as Z . A nucleus has Z protons, and $A - Z$ neutrons. The energy is the difference in mass between the nucleus and its components

$$E_B = [Zm_p + (A - Z)m_n - m_{\text{nuc}}]c^2 \quad (6.189)$$

which is the binding energy. This energy is liberated if the nucleons fuse.

6.3.2.1 Atomic mass unit

The mass of the proton is $m_p = 1.6726 \times 10^{-24}$ g, and the mass of the neutron is slightly higher, $m_n = 1.6749 \times 10^{-24}$ g. Being the building blocks of nuclei, it makes sense to make a unit out of one of these to simplify the calculations. The system in use makes ^{12}C a standard, making the atomic mass unit 1/12 of that atom, for historical reasons¹. The choice sets the atomic mass unit as $1 \text{ amu} \approx 1.6605 \times 10^{-24}$ g, which is a little lower than the proton mass, $m_p \approx 1.6726 \times 10^{-24}$, or 1.0073 amu. The neutron mass is 1.6749×10^{-24} g, or 1.0087 amu.

6.3.2.2 Fusion

A nucleus of helium-4 (^4He) is composed of two protons and two neutrons. The mass of the proton is 1.6726×10^{-24} g, and the mass of the neutron is 1.6749×10^{-24} g. One would expect the mass of the Helium nucleus to be 6.695×10^{-24} g, but it is in fact 6.646×10^{-24} g, a difference of about 0.7%.

¹John Dalton proposed in 1803 that the hydrogen mass (essentially that of the proton) be the atomic mass unit. Yet, throughout the 19th century, several different scales were proposed, and the one based on oxygen became popular. Oxygen was chosen because its abundance and reactivity, forming compounds with many other elements, and thus simplifying the determination of their atomic weights. Oxygen was assigned the atomic weight 16, which was the lowest number that did not give hydrogen a weight lower than 1. The atomic mass unit was thus set to 1/16 the mass of oxygen. In 1929, however, isotopes of oxygen were discovered, which complicated the picture. For chemists the situation did not change: dealing with large numbers of oxygen atoms in their reagents, they simply assigned the atomic mass unit to be 1/16 of the average mass of oxygens in their reagents. As long as the relative abundances of isotopes in nature was the same, it did not change their calculations. Physicists, on the other hand, dealt with single atoms, not reagents, and thus preferred a system based on the mass of one particular isotope, not a mixture. The most common oxygen isotope was chosen (^{16}O), and thus physicists and chemists used different atomic mass units for some time. The unification came late, in 1956, when ^{12}C was adopted as unit, as a compromise between chemists and physicists. Using the old physics unit based on ^{16}O was undesirable for chemists as it would change their atomic weights by 275 parts per million. Using ^{12}C would change it by only 42 parts per million, which was judged reasonable. Physicists did not object to ^{12}C as unit as it was already used as a standard mass in spectroscopy.

One would expect the mass of Helium nucleus to be 4.03188 amu, but it is in fact a little less, around 4.00153 amu, a difference of 0.03 amu, or 0.7%.

The binding energy of a Helium nucleus is thus

$$E_B \approx 0.03 \text{ amu } c^2 \quad (6.190)$$

$$\approx 4.5 \times 10^{-5} \text{ erg} \quad (6.191)$$

6.3.2.2.1 Electron-volt

This is a disturbing result. Remember an erg is the work done by a force of one dyne over the length of one centimeter. The dyne itself is the force that results in an acceleration of 1 cm/s^2 for one gram of mass.

One *gram* of mass. The length of one *centimeter*. These are everyday macroscopic, human scale, units.

One gram of material has of the order of 10^{23} atoms, more than the total number of stars in the known Universe. And yet, the binding energy of a mere 10,000 helium atoms is enough to steadily accelerate this huge amount of atoms over human-scale distances. The binding energy of a single helium atom can accelerate 10^{18} (a quintillion) other atoms to 1 cm/s^2 and sustain that acceleration for 1 cm.

4.5×10^{-5} erg is a huge amount of energy for a single atom!

And yet it looks small, written in a human scale energy like the erg.

A more insightful unit in particle physics is the electron-volt, which is the energy that an electron acquires in an electric field of 1V. The volt is an SI unit, equal to the work of one joule done on a charge of one coulomb. The coulomb is too high a charge for everyday purposes, since it is derived from the ampère, unit of current, which presupposes many charges. In Gaussian units, the unit of electric charge is the franklin (Fr), defined as the charges that, separated by 1 cm, experience the force of 1 dyne. The conversion is $1\text{C} \approx 3 \times 10^9 \text{Fr}$. The electron charge is $e \approx 4.8 \times 10^{-10} \text{Fr}$. So, one electron-volt in erg is

$$1 \text{ eV} \equiv e \times \frac{1\text{J}}{1\text{C}} \quad (6.192)$$

$$\approx (4.8 \times 10^{-10} \text{Fr}) \times \left(\frac{10^7 \text{erg}}{3 \times 10^9 \text{Fr}} \right) \quad (6.193)$$

$$\approx 1.6 \times 10^{-12} \text{ erg} \quad (6.194)$$

The binding energy of a ^4He nucleus is thus 6.6 MeV. Because of mass-energy equivalency, eV/c^2 is a unit of mass. The c^2 is usually dropped, and eV used as unit of mass, the c^2 factor implicit. The mass of the electron is 0.511 MeV, the mass of the proton is 938.28 MeV, and the mass of the neutron is 939.57 MeV.

6.3.2.2.2 Coulomb barrier

A nucleus is composed of protons and neutrons, the protons having positive charge they strongly repel each other electrically, and the nucleus is held together by the residual strong force. The strong force is the force that binds the quarks together in a hadron. The force is felt a little beyond the radius of the hadrons, given by their de Broglie wavelength

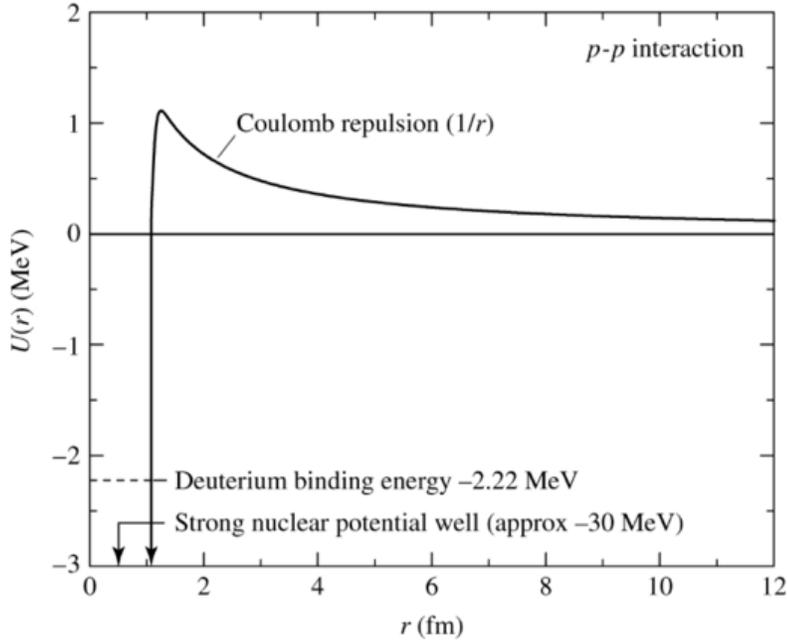


Figure 6.13: The Coulomb barrier. As two protons approach each other, their electric repulsion increases. At the distance of 10^{-13} cm, about the size of the nuclei, the repulsion is about 1 MeV strong. At the temperatures of the solar core, 10^7 K, the kinetic energy kT of a proton is of the order of keV. Classically, protons should not fuse in the core of the Sun. Quantum tunneling is necessary to overcome the Coulomb barrier.

$$\lambda \equiv \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2}mc} \approx 10^{-13} \text{ cm} \quad (6.195)$$

for a proton. This residual force is enough to bind them together in a nucleus, much in the same way that molecules are held together by a residual of the electromagnetic interaction, called London force, that binds the atom. The nuclear force decays exponential and is only felt up to a couple nuclear radii. If free protons are to fuse, they must come close together by approximately this distance.

And that poses a problem. The electric repulsion by the Coulomb force is proportional to the inverse square of the distance. At 10^{-13} cm separation, the energy of the potential is

$$E \equiv \frac{e^2}{r} \approx \frac{(4.8 \times 10^{-10} \text{ Fr})^2}{10^{-13} \text{ cm}} \approx 1 \text{ MeV} \quad (6.196)$$

That is, to fuse, two protons must overcome a Coulomb potential barrier of 1 MeV (see Fig. 6.13). At the core of the Sun, the temperature is of 10^7 K. The typical kinetic energy is $E = k_B T$, yielding energies of the order of keV ($k_B = 8.62 \times 10^{-5}$ eV/K). The

conclusion is that the kinetic energy of protons even at the core of the Sun is too low to overcome the Coulomb barrier and fuse. Classically, a nucleus of charge Z_1e and mass m_1 approaching another nucleus of charge Z_2e and mass m_2 with the energy of motion equal to E should not be able to come closer than a distance r_1 given by

$$E = Z_1 Z_2 \frac{e^2}{r_1} \quad (6.197)$$

Quantum-mechanical tunneling of the potential barrier must be taken into account so that the two nuclei can come into range of nuclear forces.

6.3.2.2.3 Quantum tunneling

The typical kinetic energy is much less than the energy of the Coulomb barrier. So, fusion must depend on the probability of tunneling through the barrier. This probability must be calculated quantum-mechanically, via the Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi = E\Psi. \quad (6.198)$$

We can re-write it as

$$\frac{1}{\Psi} \nabla^2 \Psi = -\frac{p^2}{\hbar^2} \quad (6.199)$$

with

$$p = [2m(E - V)]^{1/2} \quad (6.200)$$

For $E < V$, p is imaginary. We can use an ansatz

$$\Psi = Ae^{C/\hbar}, \quad (6.201)$$

so that $\Psi' = \Psi C'/\hbar$ and

$$\Psi'' = \Psi \left(\frac{C'^2}{\hbar^2} + \frac{C''}{\hbar} \right) \quad (6.202)$$

Because \hbar is a small number, the 1st term in the expansion is of leading order. We can thus ignore the second term and the Schrödinger equation becomes

$$C'^2 = 2m(V - E) = -p^2 \quad (6.203)$$

and thus

$$C = - \int |p| dr \quad (6.204)$$

So

$$\Psi = A \exp \left\{ -\frac{1}{\hbar} \int |p| dr \right\} \quad (6.205)$$

Notice that because the de Broglie wavelength is $\lambda \equiv h/p$, then $p/\hbar = 2\pi/\lambda \equiv k$, the wavenumber.

Given the barrier, we can write A is the incident amplitude, B is the reflected amplitude, and F is the transmitted amplitude. To the right of the barrier,

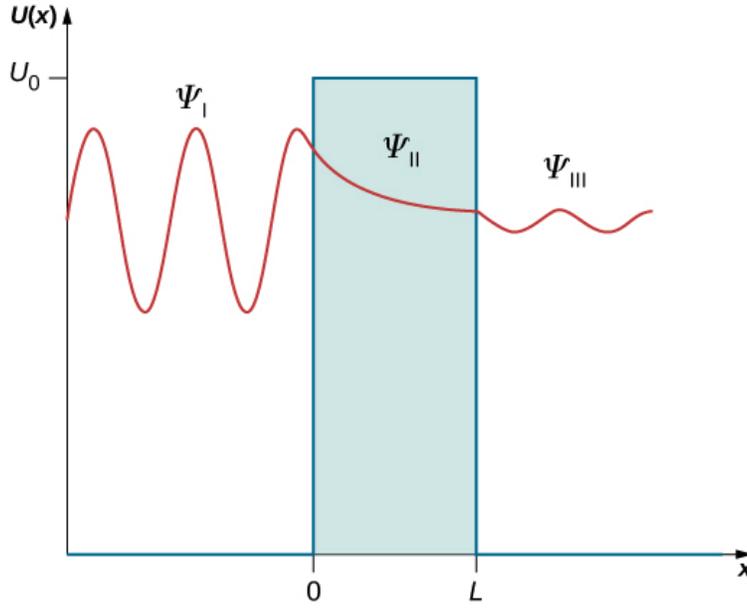


Figure 6.14: .

$$\Psi(r) = Ae^{-ikr} + Be^{ikr} \quad (6.206)$$

To the left of the barrier

$$\Psi(r) = Fe^{-ikr} \quad (6.207)$$

with $k = p/\hbar$. The transmission probability is

$$T = \frac{|F|^2}{|A|^2} \quad (6.208)$$

The non-classical region extends from r_1 to r_0 , where there is an exponential decay in amplitude. Fig. 6.14 shows the wavefunction in the region I, before the barrier, where the wave is oscillatory, in the region II, inside the barrier, where the amplitude exponentially decays, and in region III, with the transmitted wave.

$$\frac{|F|}{|A|} = \exp \left\{ -\frac{1}{\hbar} \int_{r_0}^{r_1} |p| dr \right\} \quad (6.209)$$

And because $T = (|F|/|A|)^2$, the probability is

$$P \propto T = \exp \left\{ -\frac{2}{\hbar} \int_{r_0}^{r_1} [2m(V-E)]^{1/2} dr \right\} \quad (6.210)$$

The integral can be evaluated analytically by transforming it into a trigonometrical integral with the substitution $r = r_1 \cos^2 \theta$ and $dr = -2r_1 \cos \theta \sin \theta d\theta$. The potential is thus

$$V = Z_1 Z_2 \frac{e^2}{r} = Z_1 Z_2 \frac{e^2}{r_1 \cos^2 \theta} = \frac{E}{\cos^2 \theta} \quad (6.211)$$

with E the energy of the barrier. In terms of θ , the limits are: for $r = r_1$, $\cos^2 \theta = 1$, so $\theta = 0$. For $r = r_0$, $\cos^2 \theta = r_0/r_1 \ll 1$, so $\theta = \pi/2$. The integral is thus

$$P \propto \exp \left\{ -\frac{2}{\hbar} \sqrt{2m} \int_0^{\pi/2} \left[\frac{1}{\cos^2 \theta} - 1 \right]^{1/2} (-2r_1 \cos \theta \sin \theta d\theta) \right\} \quad (6.212)$$

$$\propto \exp \left\{ -\frac{4r_1}{\hbar} \sqrt{2mE} \int_0^{\pi/2} \sin^2 \theta d\theta \right\} \quad (6.213)$$

The integral is equal to $\pi/4$, so the result is

$$P \propto \exp \left\{ -\frac{\pi r_1}{\hbar} \sqrt{2mE} \right\} \quad (6.214)$$

Substituting

$$r_1 = \frac{Z_1 Z_2 e^2}{E}, \quad (6.215)$$

we finally arrive at the expression of the probability of quantum tunneling

$$P \propto \exp \left\{ -\frac{\pi Z_1 Z_2 e^2}{\hbar} \sqrt{\frac{2m}{E}} \right\} \quad (6.216)$$

Noticing that most quantities in this expression are constants, we can write it as

$$P \propto \exp \left(-\frac{b}{\sqrt{E}} \right) \quad (6.217)$$

with

$$b = \sqrt{2m} \pi \hbar^{-1} Z_1 Z_2 e^2 \quad (6.218)$$

Eq. (6.217) is also called the *Gamow tunneling probability*, after George Gamow, who first calculated it.

6.3.2.3 Nuclear reaction rates

We want to find the quantity ε that in the luminosity equation is responsible for energy generation. The dimension of this quantity is

$$[\varepsilon] = \frac{\text{erg}}{\text{s g}} \quad (6.219)$$

that is, how much power one gets by gram of material. If we multiply it by density, we get something with units of

$$[\rho \varepsilon] = \frac{\text{erg}}{\text{s cm}^3} \quad (6.220)$$

that is, power by unit volume. We can connect this quantity to a reaction rate per unit volume, Γ , which should have units

$$[\Gamma] = \frac{1}{\text{s cm}^3} \quad (6.221)$$

So, to get $\rho\varepsilon$ one would simply need to multiply Γ by the amount of energy ΔE one gets in a single reaction, i.e.,

$$\varepsilon \equiv \frac{\Gamma\Delta E}{\rho} \quad (6.222)$$

What should go in the reaction rate Γ ? We can get an idea by dimensional analysis. The reactions are between two reagents, so it must depend on their number densities n_1 and n_2

$$\Gamma \propto n_1 n_2 \quad (6.223)$$

It must also depend on the cross section σ of the approaching nuclei

$$\Gamma \propto n_1 n_2 \sigma \quad (6.224)$$

So far the units are $\text{s}^{-1}\text{cm}^{-3}$ on the LHS and cm^{-4} on the RHS. Any missing quantities must contribute units cm/s to make the dimensionality consistent. We are missing a velocity.

$$\Gamma \propto n_1 n_2 \sigma u \quad (6.225)$$

Now, what cross section and what velocity are to be used? Particles, if in thermal equilibrium, will have a Maxwell-Boltzmann distribution of velocities. The cross sections for particles are not their physical radii but their de Broglie wavelengths, and augmented by tunneling probabilities, both functions of energy. So, we need to rely on statistical averages. We can substitute the proportionality by an equality by defining the average of these quantities

$$\Gamma = n_1 n_2 \langle \sigma u \rangle \quad (6.226)$$

where the average is an average in energy

$$\langle \sigma u \rangle = \int_0^{\infty} \sigma(E) u f(E) dE \quad (6.227)$$

and $f(E)$ is the probability distribution of energies. In thermal equilibrium, $f(E)$ is the Maxwell-Boltzmann distribution. In terms of velocity it is

$$f(u) du = \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{mu^2}{2k_B T} \right) 4\pi u^2 du \quad (6.228)$$

so in terms of energy

$$f(E) dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(k_B T)^{3/2}} \exp\left(-\frac{E}{k_B T} \right) dE \quad (6.229)$$

Thus, we only need to know the reaction cross-section $\sigma(E)$ to calculate the reaction rates. The reaction cross section should be the geometric cross-section $\pi\lambda^2$ (where

λ is the de Broglie wavelengths of the nuclei), augmented by the probability of tunneling the Coulomb barrier

$$\sigma(E) = \sigma_C \times P \quad (6.230)$$

$\sigma_C = \pi\lambda^2$ is the “classical” cross-section. Since $\lambda \equiv h/p = h/\sqrt{2mE}$, then $\lambda^2 \propto 1/E$. Given $P \propto \exp(-b/\sqrt{E})$, the energy dependency of the cross section is

$$\sigma(E) \propto \frac{1}{E} \exp\left(-\frac{b}{\sqrt{E}}\right) \quad (6.231)$$

We can substitute the proportionality by a constant

$$\sigma(E) = \frac{S}{E} \exp\left(-\frac{b}{\sqrt{E}}\right) \quad (6.232)$$

That would be the end of story, except that experiments show that S is not a constant, but a slow (i.e., weakly dependent) function of energy. So,

$$\sigma(E) = \frac{S(E)}{E} \exp\left(-\frac{b}{\sqrt{E}}\right) \quad (6.233)$$

$S(E)$ is a slow function of energy, except that occasionally it can spike. Spikes occur near resonances, where reactions are stimulated.

Substituting the cross section into Eq. (6.227), and substituting $u = \sqrt{2E/m}$

$$\langle \sigma u \rangle = \frac{2^{3/2}}{\sqrt{\pi m}} \frac{1}{(k_B T)^{3/2}} \int_0^\infty S(E) e^{-E/k_B T} e^{-b/\sqrt{E}} dE \quad (6.234)$$

The first exponential is the Maxwell-Boltzmann factor, and it decreases with energy. The second is the Gamow probability factor, and it increases with energy. Where their product is maximum, the *Gamow peak*, the reaction rate should be maximized (Fig. 6.15). Call E_0 the energy of the Gamow peak. Replace $S(E)$ by $S(E_0)$ and remove it from the integral. Then

$$\langle \sigma u \rangle = C \int_0^\infty e^{g(E)} dE \quad (6.235)$$

where

$$g(E) = -\frac{E}{kT} - \frac{b}{\sqrt{E}} \quad (6.236)$$

The energy E_0 of the Gamow peak is found by requiring $dg/dE = 0$, yielding

$$E_0 = \left(\frac{1}{2} b k T\right)^{2/3} \quad (6.237)$$

It will be left as an exercise to show that

$$g(E_0) = -\frac{3E_0}{kT} \quad (6.238)$$

And thus the average of velocity and cross section is

$$\langle \sigma u \rangle = \frac{S(E_0)}{T^{2/3}} \exp\left[-3 \left(\frac{e^4 \pi^2 m Z_1^2 Z_2^2}{2k\hbar^2 T}\right)^{1/3}\right] \quad (6.239)$$

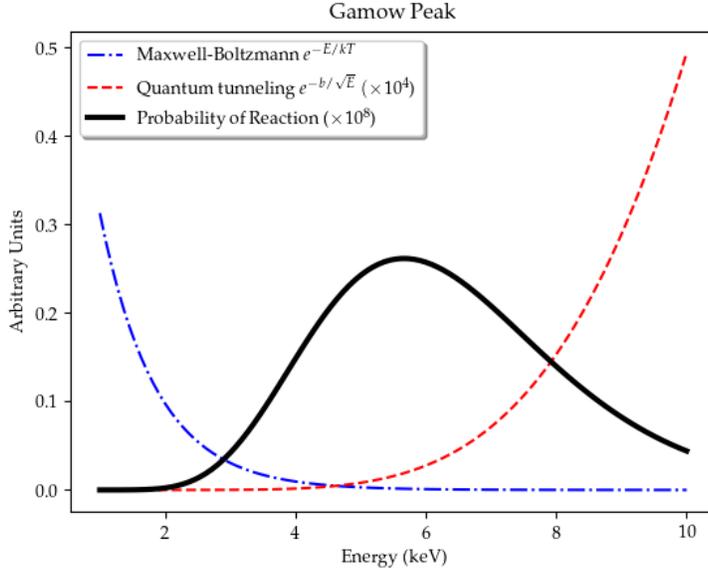


Figure 6.15: The Gamow peak for nuclear reactions. The probability of finding a particle that is too energetic compared to the Maxwell-Boltzmann average declines exponentially with energy. On the other hand, the probability of the particle being able to tunnel through the barrier increases with energy. Combined, they lead to a peak where the probability of reaction is maximized.

and the energy generation rate per unit volume

$$\Gamma \Delta E = \rho \epsilon = n_1 n_2 \langle \sigma u \rangle \Delta E \quad (6.240)$$

we can define also the concentration X , as the mass fractions of nuclei, thus $n_i = \rho X_i / m_H$. Placing $S(E_0)$ and ΔE in a constant C , we arrive at the final functional form of the nuclear burning rate

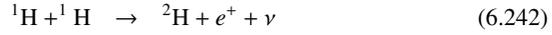
$$\epsilon = C \rho X_1 X_2 \frac{1}{T^{2/3}} \exp \left[-3 \left(\frac{e^4 \pi^2 m Z_1^2 Z_2^2}{2k\hbar^2 T} \right)^{1/3} \right] \quad (6.241)$$

6.3.2.4 Nuclear reactions

We see that the nuclear burning rate increases sharply with temperature, given the $\exp[-1/T^{1/3}]$ dependency, tapering towards an asymptote. Also, because of the dependency on mass and charge, heavier nuclei and more charged nuclei are less likely to burn than lighter, less charged, nuclei.

6.3.2.4.1 Proton-proton chain

Two nuclear reactions are of prominence importance in stars. One is the proton-proton chain. The series of reactions is



The rate is determined by the slowest reaction. The 1st reaction is the slowest, because it is mediated by the weak force as it involves the inverse beta decay of a proton into a neutron and a positron. The rate is then found by summing all energies, and dividing by the time rate of the slowest.

$$\epsilon_{pp} = 2.4 \times 10^6 \rho X^2 \left(\frac{10^6 \text{ K}}{T} \right)^{2/3} \exp \left[-33.8 \left(\frac{10^6 \text{ K}}{T} \right)^{1/3} \right] \frac{\text{erg}}{\text{s g}} \quad (6.245)$$

The $S(E)$ value is found in the laboratory in MeV, where the Coulomb barrier can be neglected, and extrapolated to keV.

6.3.2.4.2 CNO cycle

The other important nuclear reaction is the CNO cycle, for which the series of reactions is



The slowest reaction is the production of oxygen-15. The nuclear burning rate is

$$\epsilon_{CNO} = 8.7 \times 10^{27} \rho X_{CNO} X \left(\frac{10^6 \text{ K}}{T} \right)^{2/3} \exp \left[-152.3 \left(\frac{10^6 \text{ K}}{T} \right)^{1/3} \right] \frac{\text{erg}}{\text{s g}} \quad (6.252)$$

where $X_{CNO} = X_C + X_N + X_O$. The proton-proton chain dominates up to about 15 million K, after which the CNO cycle becomes the dominant source of energy production.

It is informative to evaluate ϵ for the proton-proton chain and for the CNO cycle using values appropriate to the center of the Sun: $\rho \approx 150 \text{ g cm}^{-3}$, $T \approx 1.5 \times 10^7 \text{ K}$, $X = 0.74$, $X_{CNO} = 0.02$. This gives

$$\epsilon_{pp} = 82 \text{ erg s}^{-1} \text{ g}^{-1} \quad (6.253)$$

$$\epsilon_{CNO} = 6.4 \text{ erg s}^{-1} \text{ g}^{-1} \quad (6.254)$$

Thus the proton-proton chain dominates in the Sun by about a factor of 10. However, it is important to notice that, because it has 152 instead of 33.8 in the exponential,

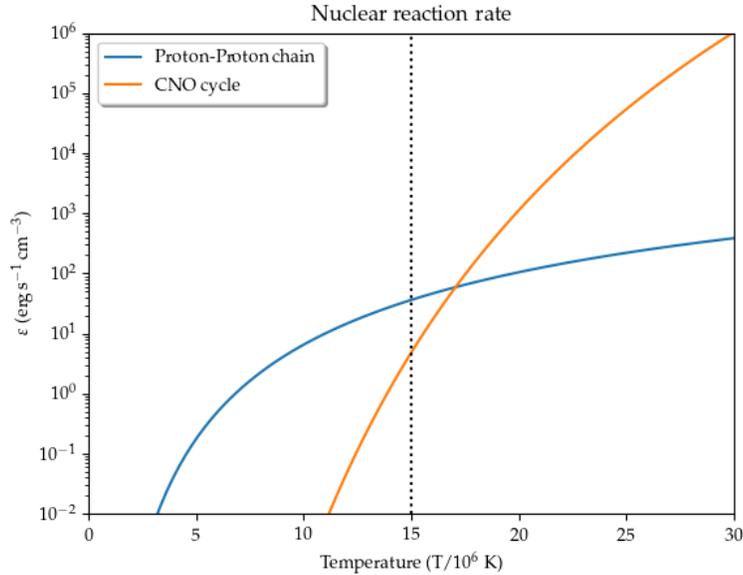


Figure 6.16: Nuclear burning rates of the proton-proton chain and the CNO cycle. The crossover is at around 18 million K. The temperature at the core of the Sun is marked by the dashed line. At the solar temperature the proton-proton chain dominates the energy production by an order of magnitude. The temperature dependency of the proton-proton chain and the CNO cycle in this temperature range can be approximated by T^4 and T^{20} , respectively.

the CNO cycle is much more temperature-sensitive than the proton-proton chain. For stars a bit more massive than the Sun, which we will see have higher central temperatures, the CNO cycle dominates. In stars smaller than the Sun, the CNO cycle is completely irrelevant.

This also brings out a general feature of nuclear reactions: the temperature-sensitivity is determined by the exponential in Eq. (6.241), which in turn depends on the charges of the nuclei involved, Z , because it is determined by the Coulomb barrier. The stronger the nuclear charge, the stronger the Coulomb barrier, and thus the higher the ignition temperature and the more temperature-sensitive the reaction becomes. The CNO cycle is more temperature-sensitive than proton-proton, and the sensitivity only gets stronger as we march up the periodic table.

6.3.2.4.3 Triple Alpha

The proton-proton chain and CNO cycles are ways to produce helium. The next step in climbing up the periodic table would be perhaps adding a hydrogen to a helium nucleus, which should produce a nuclide of atomic mass 5.



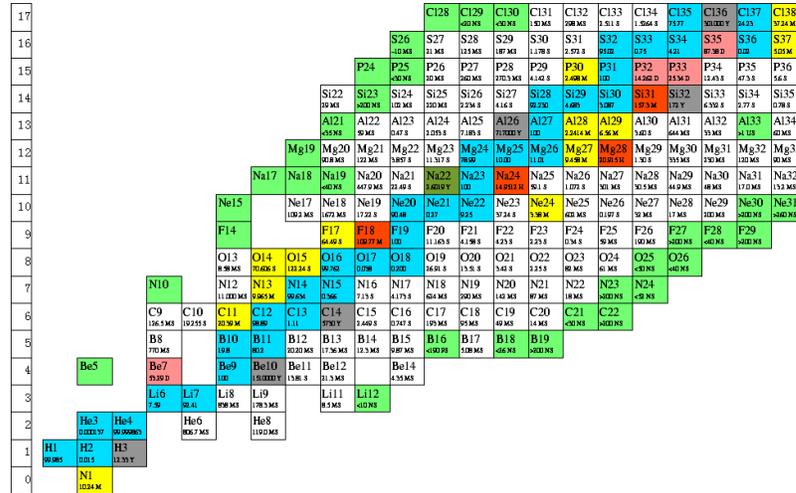


Figure 6.17: Nuclide chart. Number of protons (atomic number) increases upward, number of neutrons to the right. The blue-colored nuclides mark the stable nuclides.

Alternative, two helium atoms could fuse to produce a nuclide of mass 8



Yet, a look at a nuclide chart (Fig. 6.17) reveals no stable nuclide of mass 5 or 8. So, unless these nuclides are building blocks toward the construction of heavier nuclides, nucleosynthesis should stop here. If indeed they are building blocks to heavier nuclides, say by an unstable ${}^5\text{X}$ or ${}^8\text{X}$ nuclide combining with either a hydrogen or a helium, that reaction must occur faster than the lifetime of these unstable nuclides.

Let us see if that hypothesis holds. The unstable nuclides should exist by at least the time between encounters. This time can be estimated as $t = d_{\text{mfp}}/u$ where u is the thermal velocity of the particles, $u \approx (2E/m)^{1/2}$. The mean free path is the cubic root of the number density $n^{-1/3}$, and the number density is $n = \rho m$. For ρ of the order of 100 g/cm^3 and the mass of the proton, $n \approx 10^{26}$, and thus the mean free path is $\sim 10^{-9}$ cm. For energies of the order of keV, the velocities are of the order of 500 km/s. So, the time between encounters is

$$t_{\text{coll}} \equiv \frac{d_{\text{mfp}}}{u} \approx \frac{m^{5/6}}{\sqrt{2E}\rho^{1/3}} \quad (6.257)$$

for densities of the order of 100 g/cm^3 , energies of the order of keV, and masses of the order of the atomic mass unit, the time between encounters is $\approx 10^{-16}\text{s}$.

The nuclide that would be made by adding ${}^1\text{H}$ to ${}^4\text{He}$ would be ${}^5\text{Li}$. This is a highly unstable nucleus.

The energy of the nucleus is the rest energy of the nucleons minus the binding energy. The nucleons can only occupy discrete energy states in the nucleus, just as the electrons can only occupy discrete energy levels in the atom. If a nucleon is added when lower states are occupied, it must occupy a higher energy state. This degeneracy energy

for the case of the nuclear force can be a significant fraction of the rest energy, and thus must be taken into account in the mass of the nucleus, which is the rest energy of the nucleons, minus the binding energy, plus the degeneracy energy ². The contribution of degeneracy is mathematically equivalent to lowering the effective binding energy. At some point, adding another nucleon adds so much degeneracy energy that the effective binding energy becomes zero or positive. At this point, it becomes more energetically favorable to get rid of a nucleon, which tunnels out of the potential barrier of the nuclear force.

For neutron-heavy isotopes, this point defines a neutron drip line, beyond which isotopes are unstable. The analog process for proton-heavy isotopes defines a proton-drip line.

The ⁵Li nuclide is measured in the lab to decay into ⁴He via proton emission. Its half-life is of 3.7×10^{-22} s. That is 6 orders of magnitude lower than the time to experience an encounter. It is as if an encounter happened once a year, but this nuclide lived only for about 30 seconds.

So, none of these nuclides exist long enough to allow another reaction to occur. Climbing the periodic table by adding hydrogen to helium is not feasible. Then the other viable reaction would be to fuse helium with helium, generating beryllium.



An isotope with a half-life of 6.7×10^{-17} s, decaying into two alpha particles. The half-life is about the same as the time between collisions, so in principle the formed beryllium-8 will have time to react with a hydrogen or a helium atom and further nucleosynthesis. A reaction with hydrogen produces ⁹B, which is proton-heavy unstable, with a half-life of 8×10^{-19} s, decaying back into ⁸Be via proton emission. The reaction with helium would produce carbon



except that the energies do not match. The reaction ${}^4\text{He} + {}^4\text{He} \rightarrow {}^8\text{Be}$ can proceed because it has almost no activation energy as the energy of the ⁸Be is almost the same as that of 2 alpha particle, by about 100 keV, easily taken from the nuclides' kinetic energy. The ground state of carbon, however, is over 7.65 MeV above the combined energy of ⁸Be + ⁴He. Yet, nuclei allowed for excited discrete energy levels. An excited level of carbon may exist that allows the reaction to occur. Hoyle postulated the existence of an excited carbon state, at around the right energy, not yet observed. In nuclear physics parlance, these are called *resonances*. Effectively, they are “nuclear spectral lines”, i.e., transitions in the discrete energy levels of the nucleus. It is a “resonance” between the energy of the incoming collision, and the differences in the energy levels of the resulting nucleus.

Salpeter had arrived at the same conclusion, but balked from making the prediction of an undiscovered resonance. Yet, within a short time after Hoyle's prediction, the resonance was discovered in the lab, right at the energy level that Hoyle needed. The series of reactions leading to carbon production are thus

²Notice that the same degeneracy energy exists for electrons in an atom as well, but the energy levels of the electromagnetic interaction are negligible in comparison with the rest masses of the particles involved.



(6.263)

Where the star superscript represents an excited state. The first reaction takes ≈ 100 keV of energy, which is given by the energy of the collision. The last reaction, the decay of the excited nucleus, liberates 7.367 MeV. The whole reaction can be summarized as



which renders it the name *triple alpha*. The nuclear burning rate of the triple alpha process is

$$\varepsilon_{3\alpha} = 5.1 \times 10^8 \rho^2 Y^3 \left(\frac{10^8 \text{ K}}{T} \right)^3 \exp \left[-44 \left(\frac{10^8 \text{ K}}{T} \right) \right] \frac{\text{erg}}{\text{s g}} \quad (6.265)$$

here $Y = X_{\text{He}}$ is the abundance of Helium. Ignited at about 10^8 K, the triple-alpha reaction is the main form of carbon production in stars. Salpeter, ignoring the 7.65 ${}^{12}\text{C}$ resonance, predicted that the triple alpha could happen, but at a higher temperature, of 2×10^8 K. Yet, at this temperature, most of the produced carbon would immediately capture an alpha particle and fuse to form oxygen. Climbing up the periodic table would occur, but carbon would be as rare in the Universe as beryllium. The existence of the Hoyle resonance allows for carbon production at a relatively low temperature, avoiding carbon burning. Examining the energy levels of ${}^{16}\text{O}$ reveals the presence of a level just below the combined energy of ${}^{12}\text{C}$ and an alpha particle. Because it is below, and not above, fusion of the triple-alpha produced carbon into oxygen at this energy level does not occur.

6.3.2.5 Changes in the core

As hydrogen is converted into helium, the mean molecular weight of the stellar core increases. Given the equation of state

$$p = \frac{R}{\mu} \rho T \quad (6.266)$$

if μ increases, the same gas parcel at the same density and temperature will lead to less pressure, and the star will lose sustentation against gravity. The core must compress as a result.

If the density increases, the core shrinks, releases luminosity, and heats up. As the temperature increases, the region of the star that can undergo nuclear reactions increases slightly during the main sequence phase of evolution. The nuclear burning rate of the proton-proton chain goes as $\rho X T_6^4$ (where T_6 is the temperature in units of 10^6 K). The increase in temperature compensate the decrease in hydrogen mass fraction, and the star produces more energy. So, stars should become more luminous as they age in the main sequence³. The star is seen moving from point 1 to point 2 in the H-R diagram of Fig. 6.18.

³This also means that the Sun was *fainter* in its youth. Stellar evolution models predict that the young Sun was 30% less luminous than at present age. Because there is no record in Earth's climate to support a much lower temperature, this prediction is called *the faint sun paradox* and is an unsolved problem in astrophysics.

At point 2, hydrogen is gone in the core, leaving a core of helium ash. The inert helium core is isothermal. According to the luminosity equation

$$\frac{dT}{dr} \propto L = 0 \quad (6.267)$$

the isothermal core produces no luminosity and must be supported by against gravity by thermal pressure alone. The star loses sustentation against gravity and contracts. In the H-R diagram of Fig. 6.18, it moves along the track from point 2 to point 3. At point 3 the temperature in the inner layers has increased to the point that hydrogen burning occurs in a shell around the inert helium ash.

6.3.2.5.1 Shell burning

The star is now burning hydrogen again, using fuel that in the main sequence phase was simply around the core and not available for burning. As the volume of the star that generates energy is larger, and also because the temperature of the burning is hotter, more energy is released than in the main sequence stage. The star gets more luminous, the envelope expands and the temperature in the atmosphere decreases. In this phase of shell burning, the star is at the subgiant branch.

At the same time, ash from nuclear burning increases the mass of the helium core. This will continue until the helium core is too massive to support its own weight. This limit was calculated by Mario Schenberg and Subrahmanyan Chandrasekhar in 1942, and thus named the *Schenberg-Chandrasekhar limit* (which should not be confused with the Chandrasekhar limit for white dwarfs).

We can evaluate the Schenberg-Chandrasekhar limit by considering the equation of stellar structure for hydrostatic equilibrium

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \quad (6.268)$$

we can rearrange the equation so

$$4\pi r^3 \frac{dP}{dM_r} = -\frac{GM_r}{r} \quad (6.269)$$

The left-hand side can be rewritten by noticing that

$$\frac{d}{dM_r} 4\pi r^3 P = 4\pi r^3 \frac{dP}{dM_r} + 12\pi r^2 P \frac{dr}{dM_r} \quad (6.270)$$

and given the mass continuity equation $dM_r/dr = 4\pi r^2 \rho$,

$$\frac{d}{dM_r} (4\pi r^3 P) - 3 \frac{P}{\rho} = -\frac{GM_r}{r} \quad (6.271)$$

P/ρ is given by the equation of state

$$\frac{P}{\rho} = \frac{kT}{\mu m_H} \quad (6.272)$$

and integrating it from the center to the mass of the core

$$\int_0^{M_c} \frac{d}{dM_r} (4\pi r^3 P) dM_r - \int_0^{M_c} \frac{3kT}{\mu m_H} dM_r = - \int_0^{M_c} \frac{GM_r}{r} dM_r \quad (6.273)$$

In the first integral the derivative cancels the integration, so it is simply $4\pi R_c^3$. Because the core is isothermal, the second integral contains only constants, so

$$\int_0^{M_c} \frac{3kT}{\mu m_H} dM_r = \frac{3kT_c M_c}{\mu_c m_H} \quad (6.274)$$

the denominator is the mean mass of a particle in the core, so $M/\mu m_H = N$, the total number of particles in the core. So,

$$\frac{3kT_c M_c}{\mu_c m_H} = N_c \times 3kT_c \quad (6.275)$$

which given the kinetic energy of a particle $K = 3/2kT$, this is equal to $2K_c$, where K_c is the total thermal energy of the core.

The left-hand side is the gravitational energy of the core

$$- \int_0^{M_c} \frac{GM_r}{r} dM_r = U_c \quad (6.276)$$

So, the core pressure is

$$P_c = \frac{1}{4\pi R_c^3} (2K_c + U_c) \quad (6.277)$$

Notice that the left-hand side is the virial theorem. If we integrate from the center to the surface, where $P = 0$, we should recover the original form. Substitute now the kinetic energy given by Eq. (6.275) and the potential energy given by

$$U_c = -\frac{3}{5} \frac{GM_c^2}{R_c} \quad (6.278)$$

so

$$P_c = \frac{3}{4\pi R_c^3} \left(\frac{M_c k T_c}{\mu_c m_H} - \frac{1}{5} \frac{GM_c^2}{R_c} \right) \quad (6.279)$$

the first term is thermal energy, the second one is gravity. The maximum pressure is found by differentiating in mass, $dP/dM = 0$, which yields the radius

$$R_c|_{P_{\max}} = \frac{2}{5} \frac{GM_c \mu_c m_H}{kT_c} \quad (6.280)$$

Which yields the maximum core pressure

$$P_{\max} = \frac{375}{64\pi} \frac{1}{G^3 M_c^2} \left(\frac{kT_c}{\mu_c m_H} \right)^4 \quad (6.281)$$

revealing that the pressure decreases with mass according to $P \propto 1/M^2$. At some point, as the core builds up helium ash, the pressure decreases too much and the core cannot support itself.

At the boundary between the envelope and the core, we set $M = M_c$ and $P = P_{\text{env}}$

$$P_{\text{env}} = \int_0^{P_{\text{env}}} dP \quad (6.282)$$

$$= - \int_M^{M_c} \frac{GM_r}{4\pi r^4} dM_r \approx \frac{G}{8\pi r^4} (M^2 - M_c^2) \quad (6.283)$$

where we used an average radius \bar{r} to remove the radius from the integral. If we now assume $M_c^2 \ll M^2$ (to be verified a posteriori) and parametrize $\bar{r}^4 = R^4/\chi$, we can write

$$P_{\text{env}} \approx \frac{GM^2\chi}{8\pi R^4} \quad (6.284)$$

with χ still to be determined. And what is the radius? From the equation of state applied to the core-envelope boundary

$$T_c = \frac{P_{\text{env}} \mu_{\text{env}} m_H}{\rho_{\text{env}} k} \quad (6.285)$$

and we can use T_c at this point because the core is isothermal. We now substitute P_{env} and parametrize $\rho_{\text{env}} = \bar{\rho}/\xi$, i.e., a fraction of the average density, then

$$R \sim \frac{\chi \xi GM \mu_{\text{env}} m_H}{6 T_c k} \quad (6.286)$$

the pressure at the boundary then is

$$P_{\text{env}} \sim \frac{6^4}{\xi^4 \chi^3} \frac{1}{8\pi} \frac{1}{G^3 M^2} \left(\frac{kT_c}{\mu_{\text{env}} m_H} \right)^4 \quad (6.287)$$

If the difference in mass between the helium ash core and the star is small, then according to the equation of hydrostatic balance the pressure at the core should not be much difference between the core at the core surface and at the center. Thus, setting $P_c \approx P_{\text{env}}$

$$\frac{M_c}{M} \approx \sqrt{\frac{375}{8}} \frac{\chi^3 \xi^4}{6^4} \left(\frac{\mu_{\text{env}}}{\mu_c} \right)^2 \approx 0.2 \chi^{3/2} \xi^2 \left(\frac{\mu_{\text{env}}}{\mu_c} \right)^2 \quad (6.288)$$

For $X = 0.68$, $Y = 0.3$ and $Z = 0.02$, $\mu_{\text{env}} = 0.63$. For a helium core, $\mu_c = 1.34$, so

$$\frac{M_c}{M} \approx 0.04 \chi^{3/2} \xi^2 \quad (6.289)$$

We are left with defining the parameters χ and ξ . The first one, χ , comes from setting the average radius for the pressure integral, $\bar{r}^4 = R^4/\chi$. If we see this average as

$$\bar{R}^4 = \frac{R^4 + R_c^4}{2} \quad (6.290)$$

and set $R_c^4 \ll R^4$, then $\chi = 2$. As for ξ , it comes from writing $\rho_{\text{env}} = \bar{\rho}/\xi$, a fraction of the average density. The average density of the Sun is $\approx 1 \text{ g/cm}^3$. The stellar shows that the density where the luminosity starts to drop is roughly that. So, $\xi \sim 1$ and thus $M_c/M \approx 0.1$. A non-linear numerical calculation solving for the stellar structure finds

$$\frac{M_c}{M} \approx 0.08 \quad (6.291)$$

Thus, the helium core will collapse if its mass exceeds 8% of the star's total mass.

At point 4 in Fig. 6.18 the helium ash left by shell burning makes the core reach the Schenberg-Chandrasekhar limit. From 4 to 5 contraction of the core releases energy, increasing the luminosity and making the star expand and redden. As hydrogen keeps burning in the shell, the shell narrows, getting denser and hotter. As the hydrogen fuel is used, the star gets less luminous, yet the core luminosity expands the outer layers,

the temperature falling. A strong temperature gradient causes the convection zone to deepen. At point 6, convection has reached deep enough that it dredges up deep photons and carries them to the surface, increasing the luminosity. The atmosphere expands and reddens, ash is brought by convection to the surface, where it can be observed. The star has swelled, becoming a red giant.

6.3.2.5.2 Degeneracy

While all these changes are happening in the interior and atmosphere, what is happening to the inert contracting helium core? The core keeps contracting, heating up, and getting denser. One of the two will set first: the temperatures will either become so high that triple-alpha will be reached, or the densities will become so high that matter will become degenerate. For low mass stars, degeneracy happens first as the core contracts.

When the density is high, the electrons are forced to occupy the lowest energy levels. But electrons are fermions and cannot all occupy the ground state. Instead, as the lower states get filled, they must be stacked into states of higher and higher energy. We define the *Fermi energy* as the kinetic energy of the highest occupied state, an energy that exists even if the system is cooled to absolute zero. The Fermi energy can be converted into a measurement of temperature, momentum, or velocity

$$T_F = \frac{E_F}{k_B} \quad p_F = \sqrt{2m_e E_F} \quad u_F = \frac{p_F}{m_e} \quad (6.292)$$

In practice we are interested in a distribution of particles, not a single or a few ones. In this case, let $n(p)$ be the momentum distribution of particles. so that $n(p)dp$ is the number density of particles per unit volume with momenta between p and dp . The number density is

$$n = \int dn = \int n(p)dp \quad (6.293)$$

If we consider a volume of length limited by the uncertainty principle, $dx \sim h/dp$, then $dn = N/dV = Nh^{-3}dp^3 = Nh^{-3}4\pi p^2 dp$. For electrons, which are fermions, this box can contain only two electrons, one with spin up and one with spin down. So, $N = 2$ and

$$n(p)dp = \frac{8\pi}{h^3} p^2 dp \quad (6.294)$$

which defines the densities of states for a fully degenerate quantum gas⁴ The distribution of momenta is given by this distribution until the last occupied state (the Fermi state), and zero above. The number density is found by integrating Eq. (6.294) until the Fermi state

$$n = \int_0^{p_F} n(p)dp = \int_0^{p_F} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_F^3 \quad (6.296)$$

which then defines the Fermi momentum.

⁴Notice that, in general, one can define a momentum distribution function $f(p)$, so that

$$n(p)dp = f(p)dp^3 = 4\pi f(p)p^2 dp \quad (6.295)$$

In that formulation, the quantum box of volume $dx^3 = h^3/dp^3$ has $f(p) = 2/h^3$ up to the Fermi level and zero otherwise. For a classic gas, $n(p)dp$ is the Maxwell-Boltzmann distribution.

$$p_F \equiv \left(\frac{3h^3 n}{8\pi} \right)^{1/3} \quad (6.297)$$

Pressure integral A particle inside this box, having momentum, will exert a pressure in the box walls. This pressure is

$$dP \equiv \frac{\text{Force}}{\text{Area}} \equiv \frac{dp}{dt dA} \quad (6.298)$$

Substituting the dp from a single particle by that of the distribution of particles $dp_{\text{tot}} = pn(p)dpdV$

$$dP = \frac{2pn(p)dpdV}{dt dA} \quad (6.299)$$

The factor of 2 because the collision is elastic. Now substituting $dV = u dt dA$, and $n(p)dp = f(p)4\pi p^2 dp$

$$dP = uf(p)8\pi p^3 dp \quad (6.300)$$

So, considering a cube of six faces dA , one side of the cube will experience only a sixth of the total pressure. So,

$$P = \int dP = \frac{4\pi}{3} \int uf(p)p^3 dp \quad (6.301)$$

if $p = \gamma mu$, then we can write

$$u = \frac{p}{m\gamma} = \frac{pc^2}{E} \quad (6.302)$$

and then the pressure integral with $f(p) = 2/h^3$ is

$$P = \frac{8\pi}{3h^3} \int_0^{P_F} up^3 dp \quad (6.303)$$

$$= \frac{8\pi}{3h^3} \int_0^{P_F} \frac{p^4 c^2}{E} dp \quad (6.304)$$

The energy has the limits

$$E = \sqrt{p^2 c^2 + m^2 c^4} \approx \begin{cases} mc^2 & \text{for non-relativistic} \\ pc & \text{for fully relativistic} \end{cases} \quad (6.305)$$

So the integral gives

$$P = \begin{cases} \frac{8\pi}{15h^3 m_e} p_F^5 & \text{for non-relativistic} \\ \frac{2\pi c}{3h^3} p_F^4 & \text{for fully relativistic} \end{cases} \quad (6.306)$$

Given the definition of the Fermi momentum

$$P = \begin{cases} \frac{3^{5/3}}{15(8\pi)^{2/3}} \frac{h^2}{m_e} n^{5/3} & \text{for non-relativistic} \\ \left(\frac{3}{8\pi} \right)^{1/3} hc n^{4/3} & \text{for fully relativistic} \end{cases} \quad (6.307)$$

$n = n_e$ here is the number density of electrons. A more useful dependency would be in terms of mass density. Fully ionized hydrogen will contribute a number density of electrons equal to its own number density, $n_H = X\rho/m_H$. Helium and other species usually contribute $A/2$ electrons, because $A \approx 2Z$; so, their mass density fraction being $(1 - X)\rho$, they contribute a number density of nucleons and electrons, respectively, of

$$\frac{(1 - X)\rho}{m_H} \quad \frac{(1 - X)\rho}{2m_H} \quad (6.308)$$

The electron density is thus

$$n_e = \frac{X\rho}{m_H} + \frac{(1 - X)\rho}{2m_H} = \frac{\rho}{2m_H}(1 + X) \quad (6.309)$$

This can be written in terms of an electron mean weight

$$n_e = \frac{\rho}{\mu_e m_H} \quad (6.310)$$

where

$$\mu_e = \frac{2}{1 + X} \quad (6.311)$$

The Fermi momentum is thus

$$p_F = \left(\frac{3h^3\rho}{8\pi\mu_e m_H} \right)^{1/3} \quad (6.312)$$

and the pressure can thus be written

$$P = \begin{cases} K_1 \rho^{5/3} & \text{for non-relativistic} \\ K_2 \rho^{4/3} & \text{for fully relativistic} \end{cases} \quad (6.313)$$

The constants are

$$K_1 = \left(\frac{3}{\pi m_H \mu_e} \right)^{2/3} \frac{h^2}{20m_e m_H \mu_e} \propto \mu_e^{-5/3} \quad (6.314)$$

$$K_2 = \left(\frac{3}{\pi m_H \mu_e} \right)^{1/3} \frac{hc}{8m_H \mu_e} \propto \mu_e^{-4/3} \quad (6.315)$$

6.3.2.5.3 Helium flash

At point 7 in Fig. 6.18, the tip of the red giant branch, temperature and density have become high enough that the triple alpha reaction can happen. Yet, the core is degenerate, where the pressure does not depend on temperature. If fusion happens in an ideal gas core, the temperature increase due to nuclear reactions lead to pressure increase, and thus expansion of the gas. Expansion in turn leads to cooling and the situation is controlled fusion. If, on the other hand, fusion happens in degenerate gas, as the nuclear reactions increase the temperature, the pressure does not change as it depends only on density. The temperature increases, which increases the rate of nuclear burning, and in turn increases the temperature even further and the process runs away. The degenerate helium core is a bomb ready to explode as soon as helium fusion starts. This is known as the helium flash.

The helium flash generates about 10^{48} erg of energy, which is an incredible amount of energy, the energy that the Sun generates in 10 million years! Yet, the vast majority of this energy is used to lift the degeneracy, and nothing is seen from the outside. Helium burning then proceeds by the triple alpha reaction in a core of ideal gas. The hydrogen-burning shell is pushed outward by the new luminosity of the core, which cools it down and reduces the nuclear burning rate and thus the luminosity. The temperature gradient drops accordingly, becoming subadiabatic and the deep convection that made the star a red giant ceases. The star contracts, and settles in the horizontal branch, the analog of the main sequence for helium burning. The horizontal branch is so called because they appear at roughly constant absolute magnitude. This is because even though the spread in luminosities in the branch are roughly the same as for main sequence stars of the same mass, the average of the luminosity is much higher (RR Lyrae stars, the variable stars of the horizontal branch, have about $50L_{\odot}$). As magnitudes are a logarithmic scale of luminosity, the same degree of linear dispersion at a larger decade will appear more constant.

Once helium is exhausted, an inert core of carbon ash is left. Surrounding it, a helium-burning shell, and a hydrogen-burning shell. During shell burning, the core has zero luminosity and is isothermal. The situation is very similar to hydrogen burning phase, and the star climbs the red giant branch again. This branch is called the asymptotic giant branch.

In this final stage the star is very unstable, having thermal pulses. As the hydrogen burning layer ignites and dumps helium ash on the helium layer below, the mass of helium increases, making it slightly degenerate. When the temperature at the base of the Helium layer increases sufficiently, a helium flash occurs in the shell. The helium flash drives the hydrogen shell outwards, cooling it and turning off hydrogen burning for a while. As helium is consumed, the star contracts heating up, the hydrogen shell ignites again, dumps helium ash on the helium layer, and the process repeats. The thermal pulses are essentially a series of helium flashes. Eventually, as the star heats up, the last helium flashes produce enough energy to blow up the star, ejecting the planetary nebula.

6.3.2.6 White dwarfs

What is left after the planetary nebula is the inert core of carbon and oxygen ash, left behind from Helium burning. This core, known as *white dwarf*, is supported by hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \quad (6.316)$$

except that now the equation of state is that of a degenerate gas, with electron pressure providing support.

Making the approximation of constant density, $M_r = 4\pi r^3\rho/3$, and thus

$$\frac{dP}{dr} = -\frac{4\pi G\rho^2}{r^3} \quad (6.317)$$

we can integrate this equation from $P = 0$ at the surface until an arbitrary radius r inside the white dwarf

$$P(r) = \frac{2}{3}\pi G\rho^2 (R^2 - r^2) \quad (6.318)$$

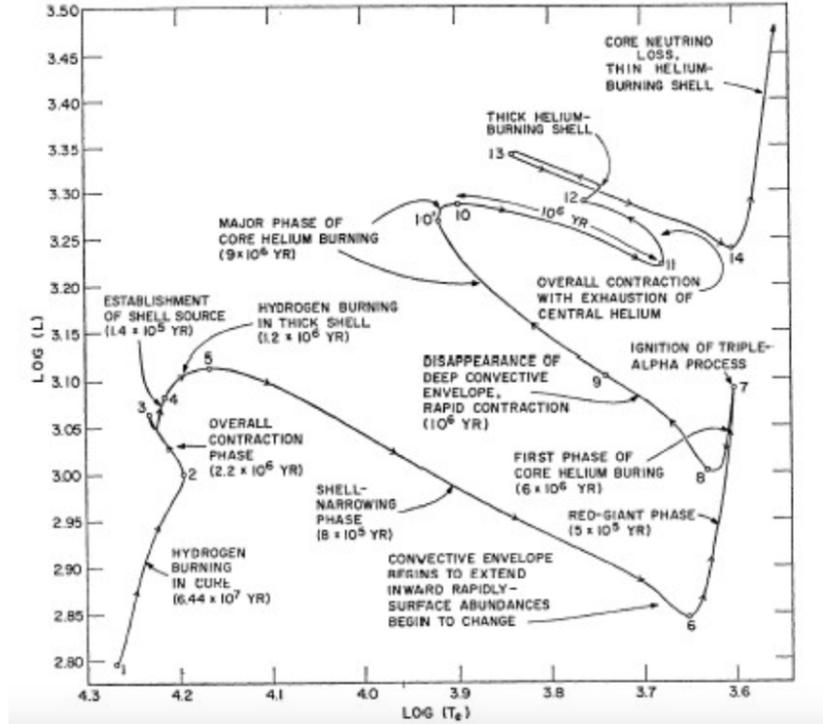


Figure 6.18:

From this it results that the pressure at the center ($r = 0$) is

$$P_c = \frac{2}{3}\pi G\rho^2 R^2 \quad (6.319)$$

6.3.2.6.1 Chandrasekhar limit

In the non-relativistic limit, the equation of state is $P = k_1\rho^{5/3}$. We can equate it with the pressure found in Eq. (6.319)

$$\frac{2}{3}\pi G\rho^2 R^2 = k_1\rho^{5/3} \quad (6.320)$$

and solve for the radius

$$R^6 = \left(\frac{3k_1}{2\pi G}\right)^3 \frac{1}{\rho} \quad (6.321)$$

Substituting $1/\rho = 4\pi R^3/3M$ for constant density

$$R^3 M \equiv \text{const} \quad (6.322)$$

Since R^3 is proportional to the volume, we arrive at the mass-volume relation for white dwarfs

$$MV \equiv \text{const} \quad (6.323)$$

As a white dwarf gains mass, it must shrink. More massive white dwarfs are smaller. This is expected because as you add mass, the star must increase the pressure to provide support against gravity, and in a white dwarf the pressure is a function of the density. The electrons must be more closely confined to generate the larger degeneracy pressure to support a more massive star.

As the electrons are packed even more closely together, the uncertainty principle implies that their velocities are increasing

$$\Delta p^3 \sim \frac{h^3}{\Delta x^3} = h^3 n_e \quad (6.324)$$

At some point, their velocities become relativistic. If $p^3 = h^3 n_e$, with $v = p/m$, then $v = h/mn_e^{1/3}$, with $n_e = \rho/m_h$, for singly ionized species. For the typical white dwarf, $M = 1M_\odot$ and $R = 10^4$ km, resulting in $v = 0.35c$. The electrons are moving at relativistic velocities, and we need to use the relativistic equation of state for degenerate matter

For the relativistic equation of state, we have that $P = k_2 \rho^{4/3}$. Equating again this pressure to Eq. (6.319),

$$\frac{2}{3}\pi G \rho^2 R^2 = k_2 \rho^{4/3} \quad (6.325)$$

and solving for the radius

$$R^3 = \left(\frac{3k_2}{2\pi G}\right)^{3/2} \frac{1}{\rho}. \quad (6.326)$$

again substituting $1/\rho = 4\pi R^3/3M$ for constant density we find

$$R^3 = \left(\frac{3k_2}{2\pi G}\right)^{3/2} \frac{4\pi R^3}{M}. \quad (6.327)$$

In Eq. (6.327) the radius cancels out on both sides. Solving for the mass yields

$$M = 4\pi \left(\frac{3k_2}{2\pi G}\right)^{3/2} \quad (6.328)$$

$$\approx 1.44M_\odot \quad (6.329)$$

Eq. (6.329) is a very curious result. Because the radius cancels out in Eq. (6.327), there seems to be only one mass that a fully relativistic white dwarf can have. This is understood as a mass *limit*. As one adds more and more mass, the white dwarf shrinks to produce more pressure. Using the non-relativistic equation of state, a volume of zero would only be achieved at infinite mass ((Eq. 6.323)), because the non-relativistic equation of state assumes that the electrons can move faster than the speed of light, producing ever more pressure as they get squeezed together. Yet, physically, as the electrons approach the speed of limit, increasingly more energy is required to increase their velocity. A partially relativistic white dwarf has a smaller radius than the non-relativistic one. Eventually, volume zero is reached for the mass given by Eq. (6.329).

This mass limit for white dwarfs is known as the *Chandrasekhar limit*, and sets the maximum mass that a white dwarf can have.

6.3.2.7 Evolution of high-mass stars

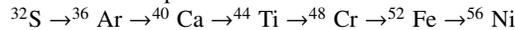
Stars of high mass, higher than 4 solar masses, end their lives in a more dramatic way than a planetary nebula. The main difference is the temperature of the core. High-mass stars have such hot cores that they never experience a helium flash. The star achieves helium burning temperatures before the gas becomes degenerate. At an intermediate mass level the carbon core becomes degenerate, and its heating to carbon burning temperature deflagrates the analog of the helium flash for carbon. Given the violence of the event, it is not known as a flash, but as the *carbon detonation*. So much energy is produced in the carbon detonation that not only the carbon degeneracy is lifted, the whole star is blown apart, leaving no remnant.

6.3.2.7.1 Climbing the alpha ladder

For even higher temperatures, carbon is burned as an ideal gas. Carbon burns into oxygen by capturing alpha particles, liberating 7.16 MeV of energy per reaction. Once carbon is exhausted, oxygen burns into neon, generating 4.73 MeV of energy per reaction. As the nuclear fuel becomes heavier, the peak in binding energy looms ever closer. The sequence of reactions is



Which goes on via capturing alpha particles. We refer to this as climbing the alpha ladder. The next steps are



Notice that capturing an alpha particle is preferred than capturing a proton because, according to the nuclide chart Fig. 6.17, increasing the proton number while keeping the neutron number constant eventually results in nuclides unstable to inverse beta decay. Conversely, capturing an alpha particle means that the nuclide is adding an equal number of protons and neutrons, thus following a path along the valley of beta stability.

6.3.2.7.2 Neutron stars

Above the Chandrasekhar limit, electron pressure cannot hold the gravitational pull. The electrons are squeezed together into the protons, and the $1.4M_{\odot}$ of material becomes completely composed of neutrons. Dividing by the mass of the neutron, the number of neutrons is

$$\frac{1.4}{m_n} = 10^{57} \text{ neutrons} \quad (6.335)$$

A single giant nucleus of $A = 10^{57}$, held together by gravity and supported by neutron degeneracy.

6.3.2.7.3 Core-collapse Supernovae

A supernova releases about 10^{53} erg or energy. 1% of this energy is in the form of kinetic energy, accelerating the interstellar medium. 0.01% of the energy is in the form of photons. The 99% is neutrinos.

Neutronization (neutron drip) Consider the following reactions

$$n \rightarrow p + e + \bar{\nu}_e \quad (6.336)$$

$$p + e \rightarrow n + \nu_e \quad (6.337)$$

The first one, beta decay, is spontaneous, the decay of a free neutron, which is mediated by the weak force, occurring with a lifetime of 13 minutes. The second reaction is *neutronization*. For neutronization to occur, energy must be given as the rest energy of the neutron exceeds the combined rest energy of the electron and the proton. The kinetic energy of a degenerate electron is the Fermi energy minus its rest mass

$$E = E_F - m_e c^2 \quad (6.338)$$

This kinetic energy must exceed the mass deficit of the reaction, i.e., the condition for the reaction to occur is

$$E \geq [m_n - (m_p + m_e)] c^2 \quad (6.339)$$

Writing the Fermi energy in terms of the Fermi momentum

$$(p_F^2 c^2 + m_e^2 c^4)^{1/2} - m_e c^2 \geq [m_n - (m_p + m_e)] c^2 \quad (6.340)$$

summing the electron rest mass on both sides and squaring the resulting expression, we find

$$p_F^2 + m_e^2 c^2 \geq (m_n - m_p)^2 c^2, \quad (6.341)$$

Isolating p_F we have

$$p_F \geq m_e c \left[\left(\frac{Q}{m_e} \right)^2 - 1 \right]^{1/2}. \quad (6.342)$$

where we substituted $Q = m_n - m_p$. We can now equate it with the definition of the Fermi momentum

$$\left(\frac{3h^3 \rho}{8\pi m_H} \right)^{1/3} \geq m_e c \left[\left(\frac{Q}{m_e} \right)^2 - 1 \right]^{1/2}. \quad (6.343)$$

and solving for the density

$$\rho \geq \rho_c \equiv \frac{8\pi m_H m_e^3 c^3}{3h^3} \left[\left(\frac{Q}{m_e} \right)^2 - 1 \right]^{3/2}. \quad (6.344)$$

This is the *critical density for neutronization*. All variables in this equation are constant. Its value is

$$\rho_c \approx 1.2 \times 10^7 \text{ g cm}^{-3} \quad (6.345)$$

When this density is exceeded, neutronization occurs. Notice also that if the gas is fully degenerate, the beta decay reaction *does not occur*, since there are no free states for the electron to occupy below the very high energy Fermi level.

At $T \approx 8 \times 10^9 \text{ K}$ and $\rho \approx 10^{10} \text{ g cm}^{-3}$ ($\approx 15M_\odot$) the electrons that gave degenerate pressure are captured by heavy nuclei and protons produced by photodisintegration. The support of degenerate pressure is gone, and the core collapses in free fall.

The collapse will continue until nuclear densities ($\rho \approx 10^{14}$) are achieved and neutron degeneracy pressure kicks in, halting the collapse. In the end, a volume the size of the Earth is compressed to 50 km.

The associated release of gravitational potential energy is *tremendous*. Since the final radius is much smaller than the original, we can approximate the release of gravitational energy as

$$W = \frac{3}{10} \frac{GM^2}{R_f} \quad (6.346)$$

For a core of 2 solar masses, $W \approx 10^{53}$ erg. *The energy of a supernova*. The implication is that supernovae are powered by the energy released when the iron core collapses into a neutron star.

The precise mechanism of the explosion is not well known, what is thought is that the sudden release of gravitational energy generates a blastwave, that triggers explosive nuclear reactions along its path. The shock violently heats and accelerates the stellar envelope. In a few hours, the shockwave reaches the surface, and from the outside, the star is seen to explode. The supernova 1987a was the first opportunity to study a supernova with modern instrumentation. The progenitor was found to have a mass of $20 M_\odot$, and neutrinos were detected 4 hours before the optical event.

A burst of neutrinos is precisely what is expected from a supernova. When the degeneracy is lifted and beta decay is allowed, the reactions



occur in loop, with beta decay followed by inverse beta decay. The neutrinos escape, carrying the energy away. This cooling process, known as Urca process, cools the core of the supernova from 10^{11} K to 10^9 K, when electron degeneracy kicks in again, inhibiting beta decay. During this cooling, the energy of the gravitational collapse was converted into 10^{53} erg worth of neutrinos, the main luminosity of the event. It is estimated that of the order of 1% of the energy of the neutrinos is deposited in the shock, that leads to the kinetic energy of the explosion. Photons account for a mere 0.01% of the energetics of a supernova event.

Supernovae are also responsible for nucleosynthesis beyond the iron peak. The high flux of neutrons allows for neutron capture that, followed by beta decay, climbs up the periodic table. The processes are classified according to the intensity of the neutron flux. If the neutron flux is such that neutron capture is slower than beta decay, it is called *s-process* (s for slow). If the neutron flux is such that neutron capture is faster than beta decay, it is called *r-process* (r for rapid). The s-process produces elements

Element	# of protons	Site
H	1	Big Bang
He, C, O	2,6,8	Big Bang + Stars
Ne-Fe	10-26	High mass stars
Co-Bi	27-83	s and r process in AGB stars, SN, NS mergers
Po-U	84-92	r process in SN (?), NS mergers

Table 6.1: Nucleosynthesis site for different elements.

up to bismuth ($Z=83$), which is the last non-radioactive element. Radioactive elements are produced by the r-process.

Observations of 1987A, however, failed to show signs of r-process elements, casting doubt on the idea that supernovae are the sites of r-elements nucleosynthesis. In 2017, EM counterpart of the gravitational wave observations of a neutron star merger was obtained, conclusively showing the presence of r-process elements. Based on this observation and the negative result from 1987A, it may be possible that r-process nucleosynthesis in the Universe is entirely dominated by neutron star mergers. The table above summarizes our current understanding of nucleosynthesis.

Problems

Analytical problems

- The solar opacity at 5000\AA is $\kappa_{5000} \approx 0.3 \text{ cm}^2/\text{g}$, and the average density of the solar photosphere is $\rho \approx 2 \times 10^{-7} \text{ g/cm}^3$.
 - How far can you see through the Sun at this wavelength? Give your answer in km.
 - How far could you see through Earth's atmosphere if it had the opacity of the solar photosphere? Consider that the density of Earth's atmosphere is $\approx 1.2 \times 10^{-3} \text{ g/cm}^3$. Give your answer in meters.
- Assume that the Sun was producing its energy by slow contraction, as suggested by Helmholtz and Kelvin.
 - Estimate the amount by which the radius of the Sun has to decrease every year to produce the observed luminosity. Give the answer in meters.
 - Calculate the lifetime of this hypothetical Sun assuming that it was born with twice its present radius, that it will "die" when it reaches essentially zero size, and constant luminosity throughout its life.
- Assume an optically thin cloud of material illuminated by a nearby luminous object.
 - By using the relationship between flux and pressure in radiative equilibrium ($K_\nu = H_\nu \tau_\nu$), show that the condition that the cloud can be ejected by radiation pressure from the nearby source is that the mass to luminosity ratio M/L for the source be less than $\kappa/(4\pi Gc)$ where G is the gravitational

constant, c is the speed of light, and κ the mass absorption coefficient of the cloud material, taken to independent of frequency (grey approximation).

- (b) Calculate the terminal velocity u attained by such a cloud under radiation and gravitational forces alone, if it starts from rest a distance R from the source. Show that

$$u^2 = \frac{2GM}{R} \left(\frac{\kappa L}{4\pi GMc} - 1 \right) \quad (6.349)$$

- (c) A minimum value for κ may be estimated for pure hydrogen as that due to Thomson scattering off free electrons, when the hydrogen is completely ionized. The Thomson cross section is $\sigma_T = 6.65 \times 10^{-25} \text{cm}^2$. The mass scattering coefficient is therefore larger than σ_T/m_H , where m_H is the mass of the hydrogen atom. Show that the maximum luminosity that a central mass M of pure hydrogen can have and still not spontaneously eject material by radiation pressure is

$$L_{\text{Edd}} = 4\pi GMcm_H/\sigma_T \quad (6.350)$$

$$= 1.25 \times 10^{38} \left(\frac{M}{M_\odot} \right) \text{erg s}^{-1} \quad (6.351)$$

4. Show that the equation for hydrostatic equilibrium can also be written in terms of the optical depth τ as

$$\frac{dP}{d\tau} = \frac{g}{\kappa} \quad (6.352)$$

where g is the gravitational acceleration.

5. Estimate the hydrogen-burning lifetimes of stars near the end of the ends of the main sequence. The lower end of the main sequence occurs near $0.08 M_\odot$, $T = 1700 \text{ K}$, and $\log(L/L_\odot) = -4.3$. An $85 M_\odot$ star near the upper end of the main sequence has an effective temperature and luminosity of $T=50\,000 \text{ K}$, and $\log(L/L_\odot) = 6$, respectively. Assume that the $0.08 M_\odot$ is entirely convective so that all of its hydrogen is available for burning rather than just its inner 10%.
6. According to current solar models, the Sun has a convection zone from $0.7R_\odot$ to the solar surface, where R_\odot is the solar radius. This convection zone contains only about 2% of the solar mass. This mass is small enough that the gravitational field in the convection zone can be taken to be $-GM_\odot/r^2$. Assuming that the temperature gradient is exactly the adiabatic gradient and show that within the convective zone the density, pressure and temperature vary as

$$\frac{dP}{dr} = -\frac{GM_\odot}{r^2} \rho \quad (6.353)$$

$$\frac{dT}{dr} = -\frac{2}{5R} \frac{GM_\odot}{r^2} \quad (6.354)$$

$$\frac{d\rho}{dr} = -\frac{3\rho^2}{5P} \frac{GM_\odot}{r^2} \quad (6.355)$$

which can be integrated to

$$T(r) = T_0 \left(\frac{r_0}{r} \right) \quad (6.356)$$

$$P(r) = P_0 \left(\frac{r_0}{r} \right)^{5/2} \quad (6.357)$$

$$\rho(r) = \rho \left(\frac{r_0}{r} \right)^{3/2} \quad (6.358)$$

7. The region of the Sun where nuclear reactions occur is only the inner 20% of the radius. That means that that region already has all the luminosity of the Sun, that is simply transported outwards. Find the energy production rate of the Sun in terms of $\text{erg s}^{-1} \text{cm}^{-3}$ in the core. What is the side of a cube in the solar core that produces 100 W?
8. Consider the Lane-Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n \quad (6.359)$$

to be solved with the boundary conditions

$$\theta = 1, \quad \frac{d\theta}{d\xi} = 0 \quad (6.360)$$

at $\xi = 0$.

- (a) Obtain the analytical solution for the case $n = 0$.
- (b) Obtain the analytical solution for the case $n = 1$. Hint: first substitute $\theta = \chi/\xi$ where χ is a new variable. Then show that this substitution transforms the Lane-Emden equation into

$$\frac{d^2\chi}{d\xi^2} = -\frac{\chi^n}{\xi^{n-1}} \quad (6.361)$$

9. The Helix nebula is a planetary nebula with an angular diameter of $16'$ that is located approximately 213 pc from Earth.
- (a) Calculate the diameter of the nebula.
- (b) Assuming that the nebula is expanding away from the central star at a constant velocity of 20 km/s, estimate its age.
10. Nuclear fusion requires binding together like-charged nuclei, so a Coulomb potential barrier has to be overcome. Consider the fusion of two protons.
- (a) Classically, the protons have to approach one another by a distance equal to their own size, of the order of 10^{-13} cm. Calculate the required gas temperature in the case that the energy required to overcome the Coulomb barrier comes from the thermal energy of the gas alone. The thermal energy is $E = 3/2 kT$, where k is the Boltzmann constant and T is the temperature. The Coulomb repulsion potential is $U = e^2/r$, where e is the electron charge, and r the distance between the protons.

- (b) The central temperature of the Sun is 1.5×10^7 K. Equate the kinetic energy of a particle in the Sun's core to the thermal energy. How many times faster than this thermal velocity do the protons need to move in order to overcome the Coulomb barrier?
- (c) The Maxwell-Boltzmann distribution of velocities is

$$n_u du = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-mu^2/2kT} 4\pi u^2 du \quad (6.362)$$

where $n_u du$ is the number of particles of mass m per unit volume having speeds between u and $u + du$. What is the fraction of total number of particles that reach the velocity you calculated in the previous problem?

- (d) Consider now that the protons can quantum-mechanically tunnel the Coulomb barrier. In this case, the particles only need to approach each other by a distance of the order of the de Broglie wavelength $\lambda = h/p$, where p is the particle's momentum and h is Planck's constant. What is the temperature needed to fuse in this case? Is that consistent with the temperature in the core of the Sun?
11. Calculate the amount of energy released or absorbed in the following reactions (express your answers in MeV):
- (a) $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{24}\text{Mg} + \gamma$
- (b) $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{16}\text{O} + 2^4\text{He}$
- (c) $^{19}\text{F} + ^1\text{H} \rightarrow ^{16}\text{O} + ^4\text{He}$

The mass of ^{12}C is 12 amu by definition, and the masses of the ^{16}O , ^{19}F , ^{24}Mg are 15.99491 amu, 18.99840 amu, and 23.98504 amu, respectively. Are these reactions exothermic or endothermic?

12. Complete the following reaction sequences, recalling that charge, lepton number and baryon number must be conserved (and that antimatter particles count as negative lepton and baryon numbers).
- (a) $^{27}\text{Si} \rightarrow ?\text{Al} + e^+ + ?$
- (b) $?^?\text{Al} + ^1\text{H} \rightarrow ^{24}\text{Mg} + 4^?\text{?}$
- (c) $^{35}\text{Cl} + ^1\text{H} \rightarrow ^{36}\text{Ar} + ?$

13. The proton-proton chain and CNO cycle both involve using four hydrogen nuclei to generate a helium nucleus, the mass difference converted into energy. Consider stars near the lower and upper ends of the main sequence. The lower end of the main sequence occurs near $0.08M_\odot$, with $\log T_{\text{eff}} = 3.23$ and $\log(L/L_\odot) = -4.3$. On the other end, a $85M_\odot$ star has an effective temperature and luminosity of $\log T_{\text{eff}} = 4.7$ and $\log(L/L_\odot) = 6.0$ respectively (the logs are base ten).
- (a) How much energy is produced in a single reaction? Give your answer in MeV.
- (b) Assume that the $0.08M_\odot$ star is entirely convective so that, through convective mixing, all of its hydrogen becomes available for burning. How many nuclear reaction events are possible for this star?

- (c) Estimate the hydrogen burning lifetime for this star.
- (d) Repeat the two above questions for the high mass star, considering that in this case, the star only burns the hydrogen in the core, which is about 10% of the total mass.
- (e) Using the information given in the problem above, calculate the radii of a $0.08M_{\odot}$ star and a $85M_{\odot}$ star. What is the ratio of their radii?
14. Consider a nucleus of charge Z_1e and mass m_1 approaching another nucleus of charge Z_2e and mass m_2 with the energy of motion equal to E . According to classical physics, the nuclei should not be able to come closer than a distance r_1 given by

$$E = \frac{Z_1Z_2e^2}{r_1} \quad (6.363)$$

- (a) Show that the tunneling probability of the two nuclei coming within range of nuclear forces is given by

$$P \propto \exp \left\{ -2 \int_{r_0}^{r_1} \left[\frac{2m}{\hbar^2} \left(\frac{Z_1Z_2e^2}{r} - E \right) \right]^{1/2} dr \right\} \quad (6.364)$$

where $m = m_1m_2/(m_1 + m_2)$ is the reduced mass and $r = r_0$ is the inner edge of the potential barrier at the nuclear surface.

- (b) Work out this integral by substituting $r = r_1 \cos^2 \theta$ and assuming $r_1/r_0 \gg 1$. Show that the final result is

$$P \propto \exp \left[-\frac{2\pi}{\hbar} \left(\frac{m}{2} \right)^{1/2} \frac{Z_1Z_2e^2}{\sqrt{E}} \right] \quad (6.365)$$

15. The maximum energy of any electron in a completely degenerate gas at $T = 0K$ is known as Fermi energy. It is given by

$$E_F = \frac{\hbar^2}{2m_e} \left[3\pi^2 \left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{2/3} \quad (6.366)$$

If the thermal energy $E_T = 3/2kT$ is less than the Fermi energy, then an average electron will be unable to make the transition to an unoccupied state, and the electron gas will be degenerate.

- (a) Assume $Z = A/2$ and show that this condition implies

$$\frac{T}{\rho^{2/3}} < 1.26 \times 10^5 \text{ K cm}^2 \text{ g}^{-2/3} \quad (6.367)$$

- (b) For the Sun, the temperature and density at the core are $T_c = 1.5 \times 10^7 \text{ K}$, and $\rho = 1.5 \times 10^2 \text{ g cm}^3$. Is the Sun's core degenerate?
- (c) Sirius B, a white dwarf, has central temperature of $T_c = 7.5 \times 10^7 \text{ K}$, and average density $\rho = 3 \times 10^6 \text{ g cm}^{-3}$. Is Sirius B degenerate?
- (d) Jupiter has a central temperature of 20000 K and density 20 g cm^{-3} . Is Jupiter's core degenerate?

16. As a white dwarf cools, it will reach a point where the electrostatic potential between neighbouring nuclei, Z^2e^2/r will be comparable to the characteristic thermal energy kT . The ratio of the two is defined to be Γ

$$\Gamma = \frac{Z^2e^2}{rkT} \quad (6.368)$$

In this expression the distance r between neighbouring nuclei is customarily defined to be the radius of a sphere whose volume is equal to the volume per nucleus. Specifically, since the average volume per nucleus is Am_H/r , r is found from

$$\frac{4}{3}\pi r^3 = \frac{Am_H}{\rho} \quad (6.369)$$

- (a) Calculate the value of the average separation r for a $0.6M_\odot$ pure carbon white dwarf of radius $0.012R_\odot$.
- (b) Much effort has been spent on precise numerical calculations of Γ to obtain increasingly realistic cooling curves. The results indicate a value of about $\Gamma = 160$ for the onset of crystallization. Estimate the interior temperature T_c at which this occurs.
17. Assume that the $1M_\odot$ core of a $10M_\odot$ star collapses to produce a Type II supernova. Assume further that 100% of the energy released by the collapsing core is converted to neutrinos and that 1% of the neutrinos are absorbed by the overlying envelope to power the ejection of the supernova remnant. Estimate the final radius of the stellar remnant if sufficient energy is to be liberated to just barely eject the remaining $9M_\odot$ to infinity. Assume that the star was a main sequence star of $4R_\odot$ and that the initial radius of the core is much larger than its final radius.
18. On February 24th 1987, a supernova was seen in the constellation of Dorado, in the vicinity of the Tarantula nebula in the Large Magellanic Cloud. Neutrinos from Supernova 1987A reached Earth travelling a distance of 55 kpc.
- (a) The neutrino flux from SN 1987A was estimated to be $1.3 \times 10^{10} \text{ cm}^{-2}$ at the location of Earth. If the average energy per neutrino was approximately 4.2 MeV, estimate the amount of energy released via neutrinos during the supernova explosion.
- (b) The neutrinos were found to have energies in the range 6-39 MeV. If the spread of 12s in arrival times was caused by neutrinos of different energies traveling at different speeds, show that the neutrino mass cannot be much more than about 20 eV.
19. Consider two point masses, each having m , that are separated vertically by a distance of 1cm just above the surface of a neutron star of radius R and mass M . Find an expression for the ratio of gravitational force on the lower mass to that on the upper mass, and evaluate this expression for $R = 10\text{km}$, $M = 1.4M_\odot$, and $m = 1\text{g}$.

An iron cube 1cm on each side is held just above the surface of the neutron star described. The density of iron is 7.86 g cm^{-3} . If iron experiences a stress (force

per cross-sectional area) of $4.2 \times 10^7 \text{ N m}^{-2}$, it will be permanently stretched; if the stress reaches $1.5 \times 10^8 \text{ N m}^{-2}$, the iron will rupture. What will happen to the iron cube? What would happen to an iron meteoroid falling toward the surface of a neutron star?

20. Work out the pressure integral

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2 c^2 + m_e^2 c^4}} dp \quad (6.370)$$

by substituting $p = m_e c \sinh \theta$ and show that the general expression for the electron degeneracy pressure given by the integral above is equal to

$$P = \frac{\pi m_e^4 c^5}{3h^3} f(x) \quad (6.371)$$

where

$$f(x) = x(2x^2 - 3) \sqrt{x^2 + 1} + 3 \sinh^{-1} x \quad (6.372)$$

and $x = p_F / m_e c$, where p_F is the Fermi momentum.

21. Estimate the density of a white dwarf for which the speed of a degenerate electron is equal to half the speed of light.
22. The Schwarzschild radius R_s is the distance from a black hole whence light cannot escape. Equate the escape velocity with the speed of light c to find

$$R_s = \frac{2GM}{c^2} \quad (6.373)$$

where M is the mass of the black hole and G the gravitational constant.

23. Combine the fundamental constants \hbar , c , and G into expressions that have units of
- Mass;
 - Time;
 - Length.

Evaluate the results numerically, in grams, seconds, and centimeters, respectively. These quantities are known as *Planck units*.

24. Find the mass for which the de Broglie wavelength is equal to the Schwarzschild radius. Give your answer in grams. Argue based on this result that the Planck mass gives the lower limit for the mass of a black hole.
25. By combining gravitation, thermodynamics and quantum physics, Stephen Hawking calculated the temperature T of a non-rotating black hole to be

$$kT = \frac{\hbar c^3}{8\pi GM} = \frac{\hbar c}{4\pi R_s} \quad (6.374)$$

where R_s is the Schwarzschild radius.

- (a) Verify that this expression has the right units.
- (b) Compute the temperature of a black hole having a mass of 1.7×10^{14} g. Give your answer in K.
- (c) Approximately what portion of the electromagnetic spectrum would this blackbody temperature correspond to?
- (d) What would the radius of a sphere having the density of water be if it had a mass of 1.7×10^{14} g. Give your answer in cm.
- (e) Compute the temperature of a $10 M_{\odot}$ black hole. Give your answer in K.
26. Stephen Hawking also found out that black holes should be evaporating. The process is due to the formation of a particle-antiparticle pair just outside the event horizon of a black hole. Usually these particles quickly recombine and disappear, but if one of the particles falls into the event horizon while its partner escapes, the recombination is avoided. The gravitational energy of the black hole was used to produce the pair, so the escaping particle is carrying away some of the black hole's mass. The net effect is the emission of particles from the black hole, known as Hawking radiation, which is accompanied by a reduction in the mass of the black hole.

- (a) Consider a black hole to be a perfectly radiating blackbody of temperature T given by Eq. (6.374). Assuming that the surface area of the black hole is $4\pi R_s^2$ where R_s is the Schwarzschild radius, show that the luminosity of the black hole due to Hawking radiation is

$$L = \frac{\hbar c^6}{15360\pi G^2 M^2} = \frac{\hbar c^2}{3840\pi R_s^2} \quad (6.375)$$

- (b) The luminosity of the black hole must originate from a loss in the black hole's internal energy. Assuming that the energy of the black hole is given by $E = Mc^2$ and that $L = dE/dt$, show that the time required for the black hole to lose all of its mass to Hawking radiation is given by

$$t_{\text{evap}} = 2560\pi^2 \left(\frac{2GM}{c^2}\right)^2 \left(\frac{M}{h}\right) \quad (6.376)$$

$$\approx 2 \times 10^{67} \left(\frac{M}{M_{\odot}}\right)^3 \text{ yr} \quad (6.377)$$

- (c) If the age of the universe is 13.7 Gyr, and primordial black holes formed in the first instant after the big bang, what is the mass of the primordial black hole that is reaching the end of its life now? Give your answer in grams.

Numerical problems

1. In the webpage of the course you will find stellar models for a massive star ($5M_{\odot}$), a solar mass star, and a $0.1M_{\odot}$ star.
- (a) For the 1 solar mass star, verify that the equations of stellar structure (for mass, pressure, luminosity and temperature) are satisfied. This may be

done by selecting two adjacent zones and numerically computing the derivatives on the left-hand sides of the equations, for example

$$dP \approx P_{i+1} - P_i \quad (6.378)$$

$$dr \approx r_{i+1} - r_i \quad (6.379)$$

- (b) For each model, make a line plot of $\nabla \equiv d \ln T / d \ln P$ as a function of $\log T$. Use an appropriate range for the temperature axis, and a range $[0, 1]$ for the ∇ axis. Then, using a different linestyle, overplot ∇_{rad} and ∇_{ad} , making sure that you label the curves appropriately.
- (c) On each graph, identify zones where convection is taking place.
2. Given Eq. (6.372),
- (a) Evaluate it numerically for various values of x and use these numerical values to make a plot of P against $\log r$.
- (b) Indicate regions of the plot corresponding to the two limiting equations and for nonrelativistic and relativistic pressures, respectively.
3. Observations of stars in the asymptotic giant branch (AGB) show that they continuously lose mass at vertiginous rates, sometimes as high as $\dot{M} \approx 10^{-4} M_{\odot}/\text{yr}$. The mechanism is not well understood (AGB stars are an active research topic!), but it seems to stem from the luminosity of these stars, coupled with the gravity on their surfaces. A popular parametrization of the mass loss rate is

$$\dot{M} = -4 \times 10^{-13} \eta \frac{L}{gR} M_{\odot} \text{yr}^{-1} \quad (6.380)$$

where L , g , and R are the luminosity, surface gravity, and radius of the star, respectively (all in solar units, $g_{\odot} = 2.74 \times 10^4 \text{ cm/s}^2$). η is a free parameter whose value is expected to be near unity.

- (a) Explain qualitatively why L , g , and R enter the equation in the way they do.
- (b) Estimate the mass loss rate of a $1 M_{\odot}$ AGB star that has a luminosity of $7000 L_{\odot}$ and a temperature of 3000 K .
- (c) Assume (incorrectly) that L , R , and η do not change with time. Derive an expression for the mass of the star as a function of time. Let $M = M_0$ when the mass loss phase begins.
- (d) Using $L = 7000 L_{\odot}$, $R = 310 R_{\odot}$, $M = 1 M_{\odot}$, and $\eta = 1$, make a graph of the star's mass as a function of time.
- (e) How long would it take for a star with an initial mass of $1 M_{\odot}$ to be reduced to the mass of the degenerate carbon-oxygen core ($0.6 M_{\odot}$)?

