# Chapter 8

# Cosmology

Cosmology is the study of the Universe as a whole. Since the Universe is everything that exists, no other discipline can claim a more encompassing subject of study.

As gravity is the only force that survives in the large scales of cosmology (even magnetic fields get feeble in comparison, since the magnetic dipole field scales at $1/r^3$), the tools of cosmology are those of general relativity. However, significant insight can be gained from a Newtonian approach, which is the one we will take. This "Newtonian cosmology" does not have too misguided a name. In fact, had Sir Isaac Newton known about the expansion of the Universe, he could have derived the main results of modern cosmology. Indeed, as we will promptly see, these results arise when *a)* the cosmological principle (see below) and *b)* conservation of energy are combined with *c)* Hubble's law.

The *cosmological principle* postulates that the Universe is homogeneous and isotropic. Of course that is not true at the scales we are used to, but at sufficiently large scales, we can expect that the distribution of galaxies in the Universe becomes isotropic and homogeneous. Fig. 8.1 shows the spatial distribution of 106688 galaxies, in the Two Degree Field (2dF) Galaxy Redshify Survey, out to about a Gpc. Althought the distribution is patchy, with clusters and voids, the distribution is essentially isotropic. We also see from the figure that the anisotropies (clusters and voids) have scales of up to 100 Mpc. At scales larger than 100 Mpc, the Universe is isotropic and homogeneous.

*Hubble's law* states that a galaxy at a distance $r$ from Earth recedes from us with velocity

$$\boldsymbol{u} = H\boldsymbol{r} \tag{8.1}$$

where $H$ is Hubble's constant. The original plot in Hubble's paper from 1929 is shown in Fig. 8.2.

If isotropy holds, all galaxies see the same Hubble law, as if they were the center. This is a straightfoward result from simple vector algebra. Consider two galaxies $A$ and $B$ away from the Milky Way from a distance $r_A$ and $r_B$, respectively. Hubble's law states that the recessional velocity of these galaxies as measured from the Milky Way is

$$\boldsymbol{u}_A = H\boldsymbol{r}_A \tag{8.2}$$

$$\boldsymbol{u}_B = H\boldsymbol{r}_B \tag{8.3}$$

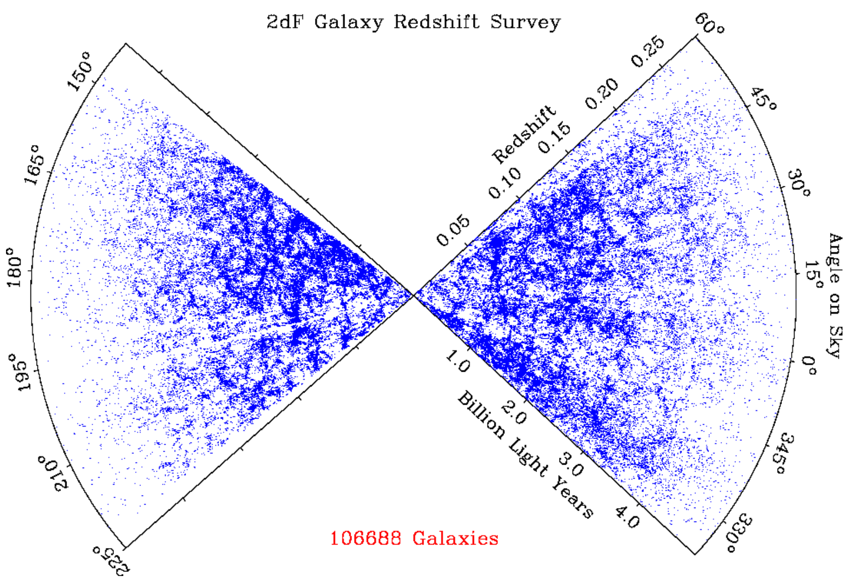


Figure 8.1: A slice of the universe, showing how $\approx 10^5$ galaxies are scattered in space down to ≈1Gpc. The anisotropies (clusters and voids) have scales of up to 100 Mpc. At scales larger than 100 Mpc, the Universe is isotropic and homogeneous. Credit: Matthew Colless.

Substracting one from the other we have the recessional velocity of galaxy $B$ as seen from galaxy $A$

$$\boldsymbol{u}_B - \boldsymbol{u}_A = H(\boldsymbol{r}_B - \boldsymbol{r}_A) \tag{8.4}$$

The equation above is equivalent to Eq. (8.1) except for a Galilean change of reference to galaxy $A$. We conclude that the observer in galaxy $A$ sees other galaxies in the Universe moving away with the same Hubble law as we do from the Milky Way Fig. 8.3.

## 8.1 Homologous expansion

Let us now consider the last principle, of conservation of energy. We first need to define to which system we will apply it. In principle any system with scales larger than the cosmological scale (≈ 100 Mpc) will be appropriate.

We thus consider a shell or large enough radius and width so that the cosmological principle holds and the shell has a homogeneous and isotropic distribution of galaxies. If we consider the Universe as composed of several of these concentric shells, each of radius $r$, the expansion is the same for all shells. That is to say, the expansion is homologous: all shells take the same time to double their radius. Therefore, we need only concentrate on one single shell to understand the behavior of the Universe.

Consider thus a shell of mass $m$ at time $t$, expanding with the Universe with recessional velocity $u = dr/dt$. As the Universe expands, the density and radius of the shell

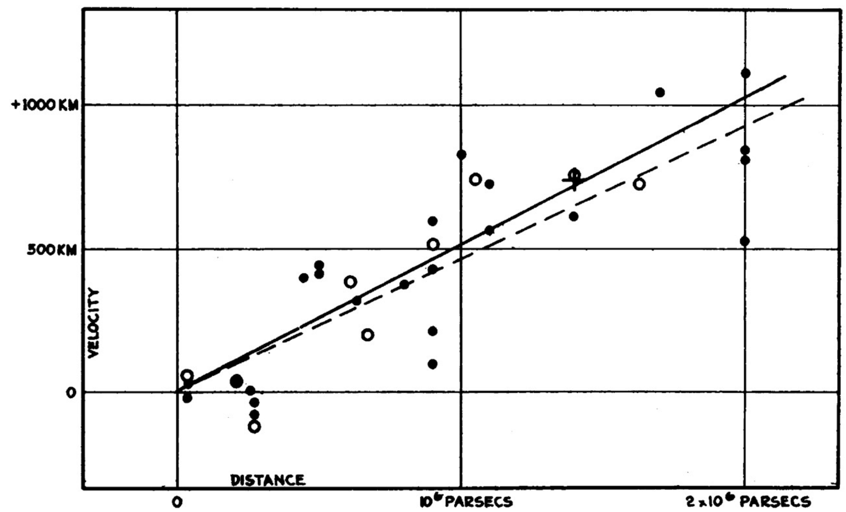


Figure 8.2: Hubble's law from Hubble's original paper in 1929. The Hubble constant is the slope of the correlation. The original caption states: "Velocity-Distance Relation among Extra-Galactic Nebulae. Radial velocities, corrected for solar motion, are plotted against distances estimated from involved stars and mean luminosities of nebulae in a cluster. The black discs and full line represent the solution for solar motion using the nebulae individually; the circles and broken line represent the solution combining the nebulae into groups; the cross represents the mean velocity corresponding to the mean distance of 22 nebulae whose distances could not be estimated individually".

change in time, i.e., $\rho = \rho(t)$ and $r = r(t)$. We can write the mechanical energy of the shell as

$$K(t) + U(t) = E \tag{8.5}$$

Given that the potential energy is the gravitational pull of the mass inside the shell, the mechanical energy is

$$\frac{1}{2}mu(t)^2 - \frac{GM_r m}{r(t)} = E \tag{8.6}$$

According to the cosmological principle, the mass $M_r$ inside the shell has the same density as the shell, so

$$M_r = \frac{4\pi}{3} r^3(t)\rho(t) \tag{8.7}$$

and we can write the mechanical energy as

$$u^2(t) - \frac{8\pi}{3} G\rho(t) r^2(t) = \frac{2E}{m} \tag{8.8}$$

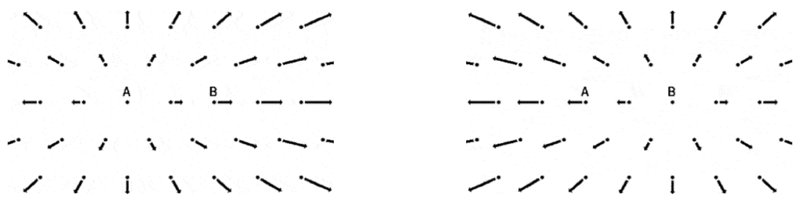


Figure 8.3: A Galilean change of reference between galaxies A and B shows that both galaxies see all other galaxies receeding from them as if they were the center of the expansion.

## 8.2 Scale factor

Since the radius $r(t)$ of each shell changes with time, it is convenient to define a reference radius against which to measure the expansion. I.e., we can write

$$r(t) = a(t)\,\omega \tag{8.9}$$

where $r(t)$ is the radius of the shell, called *coordinate distance*. The quantity $\omega$ does not change for a particular shell: it effectively "labels" a shell and follows its expansion. It is called *comoving coordinate*. $a(t)$ is dimensionless and is called the *scale factor*, i.e., the factor by which we have to scale the comoving coordinate to get the coordinate distance. By convention, at present time the scale factor is unity, $a(t_0) = 1$, corresponding to $r(t_0) = \omega$.

## 8.3 Redshift

A quantity of great interest in cosmology is the redshift, a readily observable quantity, defined as the shift in an observed wavelength $\lambda_{\rm obs}$ with respect to the original wavelength that would be emitted at rest $\lambda_{\rm rest}$

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm rest}}{\lambda_{\rm rest}} = \frac{\lambda_{\rm obs}}{\lambda_{\rm rest}} - 1 \tag{8.10}$$

Considering that the cosmological redshift is due to the Hubble flow, we can write

$$\lambda_{\rm rest} = \lambda_{\rm obs} a, \tag{8.11}$$

leading to

$$z = \frac{1}{a} - 1 \tag{8.12}$$

and conversely

$$a = \frac{1}{1+z}. \tag{8.13}$$

This nonlinear relationship between $z$ and $a$ is shown in Fig. 8.4

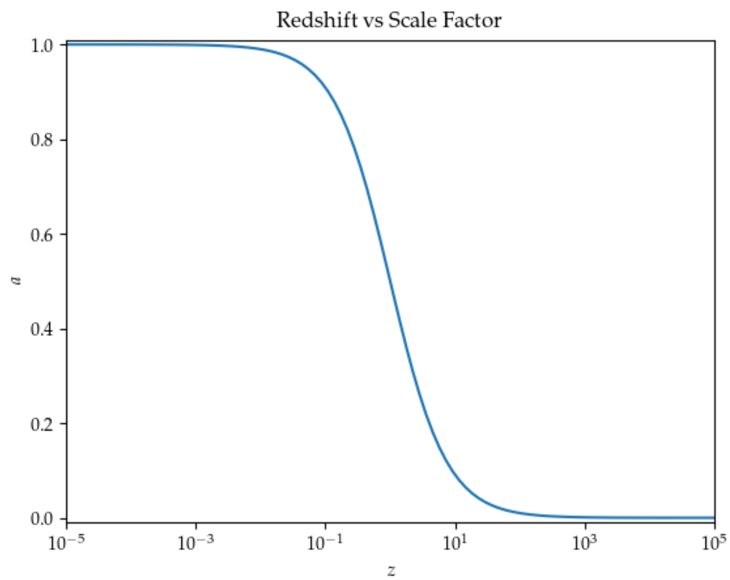


Figure 8.4: The nonlinear relationship between redshift and scale factor.

## 8.4 Friedmann Equation

The evolution of the shell (and by consequence, the Universe), is given by the time behavior of $a(t)$. We can substitute the velocity in the mechanical energy by $Hr$, as given by Hubble's law

$$u(t) = H(t)\, r(t) = H(t)\, a(t)\omega \tag{8.14}$$

or, alternatively,

$$u(t) = \frac{dr(t)}{dt} = \omega \frac{da(t)}{dt} \tag{8.15}$$

so the Hubble constant can also be written as

$$H(t) = \frac{1}{a(t)} \frac{da(t)}{dt} \tag{8.16}$$

Plugging this back in the energy equation, and omitting the "(t)" for clarity, we have

$$u^2 - \frac{8\pi}{3} G\rho r^2 = \frac{2E}{m} \tag{8.17}$$

$$\left(H^2 - \frac{8\pi}{3} G\rho\right) a^2 = \frac{2E}{m\omega^2} \tag{8.18}$$

Here we can redefine the total energy to get rid of the mass of the shell $m$, the co-moving distance $\omega$, as well as the factor 2. Considering Einstein's mass-energy equivalence, $E = mc^2$, we can set

$$E = -\frac{1}{2}m\omega^2\, kc^2 \tag{8.19}$$

where $k$ is a constant. So,

$$\left(H^2 - \frac{8\pi}{3}G\rho\right)a^2 = -kc^2 \tag{8.20}$$

and we arrive then at the Friedmann equation

$$\boxed{\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2 = -kc^2} \tag{8.21}$$

## 8.5 Closed, Open, or Flat

Based on the sign of the energy, the Universe has three behaviors:

| | |
|---|---|
| Closed (bounded) Universe: | $k > 0$ (negative energy) |
| Open (unbounded) Universe: | $k < 0$ (positive energy) |
| Flat Universe: | $k = 0$ (zero energy). |

For critical density, $k = 0$

$$\left(H^2 - \frac{8\pi G}{3}\rho\right)a^2 = 0 \tag{8.22}$$

where the equality holds when the density has the critical value

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \tag{8.23}$$

To find the numerical value of this critical density, we need to know the Hubble constant. The Hubble constant is usually written as

$$H_0 = 100\, h\ \mathrm{km\,s^{-1}\,Mpc^{-1}} = 3.24 \times 10^{-18}\, h\ \mathrm{s^{-1}}, \tag{8.24}$$

where the quantity $h$ is historical. In the early days of cosmology $H_0$ could not be measured precisely, but the order of magnitude was estimated at 100 km $s^{-1}$ $Mpc^{-1}$. So, it was written as that number, multiplied by a dimensionless factor that embodied how far from this round number the actual value was. The original estimate for $h$ was between 0.5 and 1. WMAP[1] measured $h = 0.71^{0.04}_{0.03}$, so

$$[H_0]_{\mathrm{WMAP}} = 71\ \mathrm{km\,s^{-1}\,Mpc^{-1}} = 2.30 \times 10^{-18}\mathrm{s^{-1}} \tag{8.25}$$

and the present value of the critical density is

$$\rho_{c,0} = 9.47 \times 10^{-27}\ \mathrm{kg\,m^{-3}} \tag{8.26}$$

[1] The value was updated by the Planck mission to $h = 0.678 \pm 0.0077$.

Which is roughly six hydrogen atoms per cubic meter. WMAP measured the density of visible matter in the Universe as $\rho_{b,0} = 4.17 \times 10^{-28}$ kg m$^{-3}$, or 4% of the critical density.

## 8.6 An Empty Universe

This 4% is almost completely inconsequential. If that is all the Universe has, we can ignore matter altogether and treat the Universe as empty in first approximation. We can readily solve the evolution of a Universe without matter by setting $\rho = 0$ in Eq. (8.20)

$$H^2 a^2 = -kc^2 \tag{8.27}$$

In this case $k$ is negative, so we work with $k = -|k|$

$$H^2 a^2 = |k|c^2 \tag{8.28}$$

Given Eq. (8.16),

$$\frac{da}{dt} = \sqrt{|k|}\, c \tag{8.29}$$

which is readily integrated to yield the scale factor of an empty universe as a function of time

$$a(t) = \sqrt{|k|}ct \tag{8.30}$$

The value of $k$ is given by applying Eq. (8.28) to the present time

$$H_0^2 a_0^2 = |k|c^2 \tag{8.31}$$

and given $a_0 = 1$,

$$|k| = \frac{H_0^2}{c^2} \tag{8.32}$$

Substituting into Eq. (8.30)

$$a(t) = H_0 t \tag{8.33}$$

The evolution of this empty Universe is shown in Fig. 8.5. The present time has $a = 1$, so $t = H_0^{-1}$ gives the age of this empty Universe. We define the inverse of the Hubble constant as the *Hubble time*, denoted by $t_H$

$$t_H = H_0^{-1} \tag{8.34}$$

Given the best measurement of $H_0 = 2.18 \times 10^{-18}$ s$^{-1}$, were the universe empty, it should have age $t = t_H = 13.78$ Gyr $\approx 14$ Gyr.

We cannot measure scale factor directly, but we can measure redshift. Given Eq. (8.33), the relationship between time and redshift in an empty universe is

$$t = \frac{t_H}{1+z}, \tag{8.35}$$

The age of this Universe is found by setting $z = 0$ (or conversely, $t = t_H$). At $z = 1$, $t = t_H/2$, so an object of redshift equal to one is at half the age of the Universe. The scale factor at $z = 1$ was $a = 1/2$, i.e., the Universe was also half its present size.

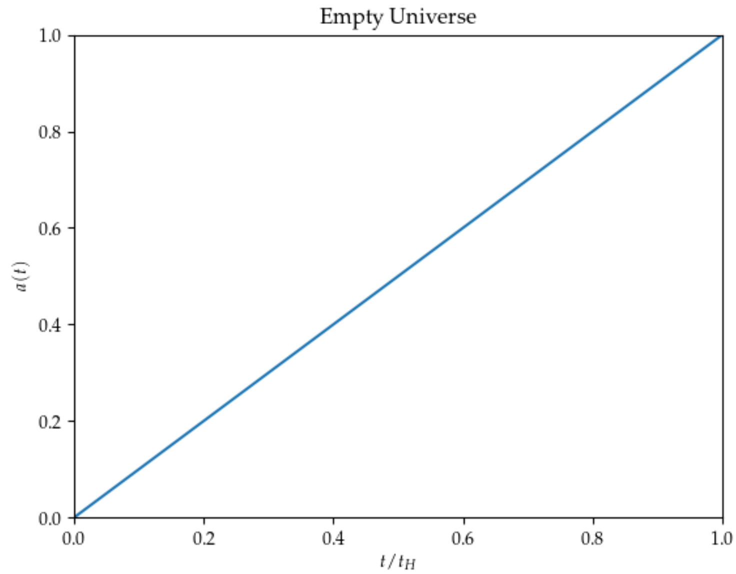


Figure 8.5: Evolution of an empty universe. The scale factor increases linearly in time as $a(t) = t/t_H$, where $t_H = H_0^{-1}$.

## 8.7 Density Parameter

Consider the ratio of measured density to critical density

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)}. \tag{8.36}$$

We can use this quantity to write the Friedmann equation compactly

$$H^2(1 - \Omega)a^2 = -kc^2. \tag{8.37}$$

Considering the sign of the sides of the equation, if

| | | |
|---|---|---|
| $\Omega > 1$ | then $k > 0$, $E < 0$ | Universe is closed; |
| $\Omega < 1$ | then $k < 0$, $E > 0$ | Universe is open; |
| $\Omega = 1$ | then $k = 0$, $E = 0$ | Universe is flat. |

Presently, the value of the density parameter is

$$\Omega_0 = \frac{\rho_0}{\rho_{c,0}} = \frac{8\pi G\rho_0}{3H_0^2} \tag{8.38}$$

and we can use the present values of the Hubble constant and $R_0 = 1$ to write for the present time the value of the constant $k$

$$k = \frac{H_0^2}{c^2}(\Omega_0 - 1) \tag{8.39}$$

According to WMAP, the density of matter (dark + luminous) is $\Omega_{m,0} = 0.27 \pm 0.04$. And the density of luminous matter alone is $\Omega_{b,0} = 0.044 \pm 0.004$. If that is all that there is in the Universe, $k$ is negative and the Universe is open.

We can write the ratio of the density parameter to the current density parameter as

$$\frac{\Omega}{\Omega_0} = \frac{\rho}{\rho_0}\frac{H_0^2}{H^2} \tag{8.40}$$

And considering the conservation of mass, $\rho/\rho_0 = 1/a^3$. Writing this in terms of the redshift,

$$\frac{\Omega}{\Omega_0} = (1 + z^3)\frac{H_0^2}{H^2} \tag{8.41}$$

or

$$\Omega H^2 = (1 + z)^3 \Omega_0 H_0^2 \tag{8.42}$$

Substituting this in the Friedmann equation, we arrive at

An interesting result arises from these equations as well. Substituting the value of $k$ from Eq. (8.39) into Eq. (8.37) and substituting scale factor for redshift ($a$ for $z$)

$$H^2(1 - \Omega) = H_0^2(1 - \Omega_0)(1 + z)^2 \tag{8.43}$$

Solving for $H$

$$H = H_0(1 + z)\left(\frac{1 - \Omega_0}{1 - \Omega}\right)^{1/2} \tag{8.44}$$

If we substitute this equation into Eq. (8.42), we find

$$\Omega = \left(\frac{1 + z}{1 + \Omega_0 z}\right)\Omega_0 = 1 + \frac{(\Omega_0 - 1)}{(1 + \Omega_0 z)} \tag{8.45}$$

A couple of results can be derived from this equation.

First, as $z \to \infty$, $H \to \infty$. That is, as the scale factor goes to zero, the rate of expansion goes to infinity. This means that if the universe had zero size, there was a big bang.

Second, the sign of $\Omega - 1$ does not change. The Universe is either always closed, always open, or always flat. The last is particularly interesting: if $\Omega = 1$ at any time, then $\Omega = 1$ at all times.

Third, as $z \to \infty$, $\Omega \to 1$. No matter if the Universe is open or closed, the early Universe was essentially flat. The last result allows us to simplify several equation in the early Universe by setting $k = 0$.

## 8.8 The Early Universe

For $k = 0$, the Friedmann equation becomes

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G\rho}{3}a^2 \tag{8.46}$$

For matter, the mass is conserved with the expansion, so $\rho/\rho_0 = 1/a^3$, so we can write

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G\rho_0}{3a} \tag{8.47}$$

This can be integrated

$$\int_0^a \sqrt{a'}\, da' = \sqrt{\frac{8\pi G\rho_0}{3}} \int_0^t dt' \tag{8.48}$$

which yields

$$a_{\mathrm{flat}} = (6\pi G\rho_0)^{1/3} t^{2/3}. \tag{8.49}$$

Since $k = 0$, the density is critical, so we simplify it by substituting

$$\rho_0 = \rho_c = \frac{3H_0^2}{8\pi G} \tag{8.50}$$

which leads to

$$a_{\mathrm{flat}} = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{8.51}$$

we also substitute the Hubble constant by the Hubble time $t_H = H_0^{-1}$. With these,

$$a_{\mathrm{flat}} = \left(\frac{3t}{2t_H}\right)^{2/3} \tag{8.52}$$

This yields the scale factor of a flat universe at all times. The evolution for this Universe is shown in Fig. 8.6

### 8.8.1 Closed and open Universe

If $\Omega \neq 1$ the Friedmann equation is solvable, though the solution is more involved. The algebra is tedious, so we simply quote the result. For a closed universe,

$$a_{\mathrm{closed}} = \frac{4\pi G\rho_0}{3kc^2}\left[1 - \cos\left(\frac{t}{t_H}\right)\right]. \tag{8.53}$$

The closed universe has infinite bounces, which are mathematical artifacts. For an open universe,

$$a_{\mathrm{open}} = \frac{1}{2}\frac{\Omega_0}{(1-\Omega_0)}\left[\cosh\left(\frac{t}{t_H}\right) - 1\right]. \tag{8.54}$$

## 8.9 Age of the Universe

These results for $a(t)$ can be inverted to yield the age of these model Universes. For the flat universe ((Eq. 8.52)), substitute $a = 1/(1+z)$ to yield

$$\frac{t(z)}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}} \tag{8.55}$$

For the current age of the Universe, set $z = 0$. So,

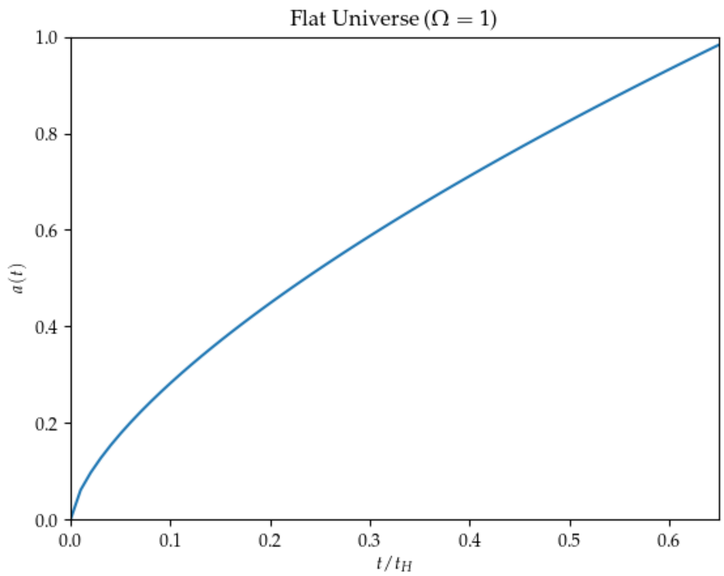


Figure 8.6: Evolution for a flat universe. The scale factor increases in time as $a(t) = (3t/2t_H)^{2/3}$, where $t_H = H_0^{-1}$. The age of this Universe (time to reach $a = 1$) is $t = 2t_H/3$, approximately 9.2 Gyr.

$$t_0 = \frac{2}{3}\, t_H \approx 9.2\,\text{Gyr} \tag{8.56}$$

The age is too low because this is the age of a Universe filled with only matter. Yet, there is more to the Universe than just matter. Photons, for instance, outnumber baryons by two billion to one. Let us include the effect of radiation in the Friedmann equation.

## 8.10 Pressure and the fluid equation

We have considered only that the shell has kinetic and potential energy, for the case of matter. In the case of radiation, or other components of the universe, matter can produce pressure. To understand how this can be taken into account in cosmology, let us apply the first law of thermodynamics

$$dU = dQ - dW \tag{8.57}$$

The Universe has the same temperature all over, so there is no heat flux, $dQ = 0$ (the expansion is adiabatic). So

$$dU = -dW = -PdV \tag{8.58}$$

And dividing by $dt$,

$$\frac{dU}{dt} = -P\frac{dV}{dt} \tag{8.59}$$

Since $V = 4\pi r^3/3$, we have

$$\frac{dU}{dt} = -\frac{4\pi}{3}P\frac{dr^3}{dt} \tag{8.60}$$

We can define the internal energy per unit volume,

$$u = \frac{U}{V} = \frac{3U}{4\pi r^3} \tag{8.61}$$

So $U = 4\pi r^3 u/3$, leading to

$$\frac{d}{dt}\left(r^3 u\right) = -P\frac{dr^3}{dt} \tag{8.62}$$

Writing $u \equiv \rho c^2$ yields

$$\frac{d}{dt}\left(r^3 \rho\right) = -\frac{P}{c^2}\frac{dr^3}{dt} \tag{8.63}$$

Now using $r = a\omega$, we obtain the *fluid equation*

$$\boxed{\frac{d}{dt}\left(a^3 \rho\right) = -\frac{P}{c^2}\frac{da^3}{dt}} \tag{8.64}$$

This equation can be used to determine the pressure of different components of the Universe or, conversely, how $\rho$ changes with time. For that, we need an equation of state. We can write

$$P = wu = w\rho c^2 \tag{8.65}$$

where $w$ is a factor to be determined for each component. We already saw that for dust (matter and dark matter), $w_m = 0$. If we plug that ($P = 0$) into the fluid equation

$$\frac{d}{dt}\left(a^3 \rho_m\right) = 0 \tag{8.66}$$

and we get $a^3\rho_m =$ const as we used in the previous solutions. In general, for pressure given by $P = w\rho c^2$ the fluid equation is

$$\frac{d}{dt}\left(a^3 \rho\right) = -w\rho\frac{da^3}{dt} \tag{8.67}$$

Adding the opposite of the RHS on both sides

$$\frac{d}{dt}\left(a^3 \rho\right) + 3wa^2\rho\frac{da}{dt} = 0 \tag{8.68}$$

Substituting $\rho a^3 = \alpha$, we have

$$\frac{d\alpha}{dt} + \frac{3w\alpha}{a}\frac{da}{dt} = 0 \tag{8.69}$$

Having $d\alpha/dt$ in one term and $\alpha$ in another, this looks like a product derivative. If we multiply the above equation by $\beta$

$$\beta\frac{d\alpha}{dt} + \frac{3w\beta\alpha}{a}\frac{da}{dt} = 0, \tag{8.70}$$

we can equate this to $(\alpha\beta)' = \beta\alpha' + \alpha\beta'$, provided

$$\frac{d\beta}{dt} = \frac{3w\beta}{a}\frac{da}{dt}. \tag{8.71}$$

Solving for $\beta$

$$\frac{d\beta}{\beta} = 3w\frac{da}{a} \tag{8.72}$$

$$\ln\beta = 3w\ln a \quad \leftrightarrow \quad \beta = a^{3w} \tag{8.73}$$

Therefore, we have

$$\beta\alpha' + \alpha\beta' = (\beta\alpha)' = 0 \tag{8.74}$$

Implying $\beta\alpha$ is a conserved quantity. Substituting $\alpha$ and $\beta$, we find a conservation form for the fluid equation

$$\boxed{\frac{d}{dt}\left(a^{3(1+w)}\rho\right) = 0} \tag{8.75}$$

This equation can be used for any fluid that we can write the equation of state as $P = w\rho c^2$. Given $a(t_0) = 1$, any such substance will follow

$$a^{3(1+w)}\rho = \rho_0 \equiv \text{const} = \rho_0 \tag{8.76}$$

For radiation, $w_{\text{rad}} = 1/3$. The evolution of radiation density as the universe expands is thus

$$\rho_{\text{rad}} = \frac{\rho_{\text{rad},0}}{a^4} \tag{8.77}$$

We understood the evolution of matter as $1/a^3$ as geometric dilution. The above equation implies that photons experience another factor $a$ decrease in their density aside from the $a^3$ factor from geometric dilution. This is understood as we realize that according to matter-energy equivalence, Eq. (8.77) can also be read as

$$u_{\text{rad}} = \frac{u_{\text{rad},0}}{a^4} \tag{8.78}$$

A factor $a^3$ is from geometric dilution from the expansion. The extra factor $a$ comes from long-wavelength photons being redshifted with the expansion, thus having less energy.

## 8.11 Temperature and the Cosmic Microwave Background Radiation

We can substitute $u \propto T^4$ in Eq. (8.78) to obtain

$$T = \frac{T_0}{a} \tag{8.79}$$

where $T_0$ is the current temperature of the Universe. This equation shows that the temperature of the Universe decreases linearly with the expansion. In terms of redshift,

$$\boxed{T = T_0(1+z)} \tag{8.80}$$

So, the Universe was hotter at earlier times than now. At some point the temperature should have been $T \approx 3000\,K$. At this temperature, the average energy of a photon is close to 13.6 eV, the ionization potential of hydrogen. Before this time, photons were energetic enough to keep the universe ionized. Protons and electrons were free, and the mean free path of the photons was small, as they were frequently being absorbed in successive ionization events. As the universe expanded and cooled below this temperature, electrons could finally bind with protons long term without photons ionizing them. We call this event *recombination*. Notice also that a consequence of recombination is that without ionization events photons now were free to travel unencumbered. Radiation and matter decoupled: the Universe became *transparent*. Photons emitted at this time should still be around, traveling across space as the Universe expands. We measure these photons as the *Cosmic Microwave Background Radiation* (CMBR), shown in Fig. 8.7: the Universe is a black body at $T_0 = 2.7$K. The CMBR was emitted at the recombination event, when the Universe was at $T \approx 3000\,K$. According to Eq. (8.80), this happened at redshift $z \approx 1000$: the CMBR was released when the Universe was a thousandth of its present size.

## 8.12 Acceleration equation

Considering again the Friedmann equation, the gravitational energy term has a $a^2\rho$ product. If we multiply it by $a$ and take the time derivative, we will have a $d/dt(a^3\rho)$ term, as it appears in the fluid equation. Let us then multiplying the whole Friedmann equation by $a$

$$\dot{a}^2 a - \frac{8\pi G}{3}\rho a^3 = -kc^2 a \tag{8.81}$$

and take the time derivative

$$\dot{a}^3 + 2\dot{a}\ddot{a}a - \frac{8\pi G}{3}\frac{d}{dt}\left(\rho a^3\right) = -kc^2\dot{a}. \tag{8.82}$$

Substituting Eq. (8.64)

$$\dot{a}^3 + 2\dot{a}\ddot{a}a + \frac{8\pi G}{3}\frac{P}{c^2}3a^2\dot{a} = -kc^2\dot{a}. \tag{8.83}$$

And substituting $kc^2$ as given by Friedmann equation

$$\dot{a}^3 + 2\dot{a}\ddot{a}a + \frac{8\pi G}{3}\frac{P}{c^2}3a^2\dot{a} = \dot{a}^3 - \frac{8\pi G}{3}\rho a^2\dot{a}. \tag{8.84}$$

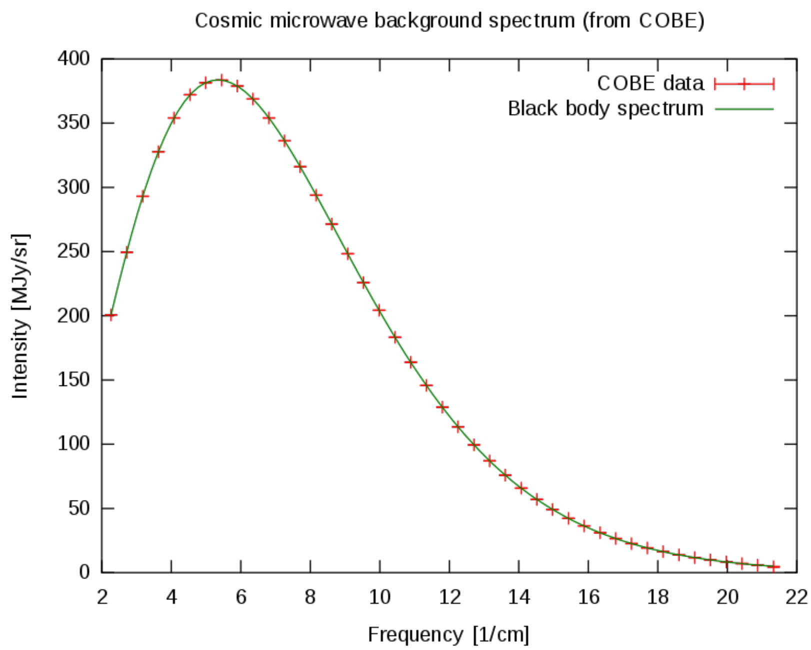


Figure 8.7: Measurement of the cosmic microwave background radiation (CMBR) shows that the Universe is essentially a black body at 2.7 K.

The $\dot{a}^3$ term cancels in both sides. Cancelling also the remaining $a\dot{a}$ in all terms, we arrive at the *acceleration equation*

$$\boxed{\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)a} \tag{8.85}$$

We see from this equation that pressure slows down the expansion (for $P > 0$). This is counter-intutitive, since we are used to pressure making thing expand. In fact, this notion does not apply in cosmology. In usual fluid mechanics, it is the pressure *gradient* that enters in the momentum equation. In cosmology there is no pressure gradient to exert force because inner and outer pressures are the same in the shell. What is happening is that due to energy-mass equivalence, the gravity of the fluid's *kinetic energy* is providing the slow-down.

## 8.13 Radiation-dominated era

In relativity there is a mass-energy equivalence; in a model with only dust, dust's gravity was slowing down the expansion. Yet, CMB photons also gravitate, so they should have some impact on the evolution of the Universe. Considering

$$P = \omega u \tag{8.86}$$

the different components of the Universe dilute differently with the expansion, according to the fluid equation

$$a^{3(1+\omega)}\rho \equiv \text{const} \tag{8.87}$$

With $w = 0$ for matter and $w = 1/3$ for radiation, these dilute as

$$\begin{aligned} a^4\rho_{\text{rad}} &= \rho_{\text{rad},0} \\ a^3\rho_m &= \rho_{m,0} \end{aligned} \tag{8.88}$$

We see that $\rho_{\text{rad}}$ increases more rapidly with $a$ than $\rho_m$ as $a$ decreases. At some point, we should arrive at a scale factor $a_{r,m}$ where $\rho_{\text{rad}}$ and $\rho_m$ have equal contributions. Earlier than this scale factor radiaton dominates, i.e., a *radiation-dominated era*. This transition from radiation era to matter era occurred when the scale factor satisfied $\rho_{\text{rad}} = \rho_m$ (or equavently $\Omega_{\text{rad}} = \Omega_m$). We can find the scale factor $a_{r,m}$ that corresponds to it from the present day values for $\Omega_{\text{rad}} = 8.24\times10^{-5}$ and $\Omega_m = 0.27$, and using Eq. (8.88) to solve for $a$. This yields

$$a_{r,m} = \frac{\Omega_{\text{rad},0}}{\Omega_{m,0}} = 3.05\times10^{-4} \tag{8.89}$$

The associated redshift is $z_{r,m} = 1/a_{r,m} - 1 = 3270$. The temperature of the Universe at this redshift, according to Eq. (8.80), was $T_{r,m} = 8920\,\text{K}$. That is, it happened before recombination. Let us now calculate the age of the Universe considering the presence of radiation. Remember that a dust-only Universe gave an age of 9 Gyr. We write the Friedmann equation with dust and radiation

$$\left(\frac{da}{dt}\right)^2 - \frac{8\pi G}{3a}\rho_{m,0} - \frac{8\pi G}{3a^2}\rho_{\text{rad},0} = 0 \tag{8.90}$$

We set $k = 0$ in the RHS because the early Universe was flat. Integrating the equation

$$\int_0^a \frac{a'da'}{(a'\rho_{m,0} + \rho_{\text{rad},0})^{1/2}} = \left(\frac{8\pi G}{3}\right)^{1/2}\int_0^t dt' \tag{8.91}$$

the integral in the LHS is of the type

$$\int \frac{x}{\sqrt{ax+b}}dx = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b} + \text{const} \tag{8.92}$$

where $a = \rho_{m,0}$, and $b = \rho_{\text{rad},0}$. In terms of these quantities the general solution is

$$\left(\frac{8\pi G}{3}\right)^{1/2} t(a) = \frac{2(\rho_{m,0}a - 2\rho_{\text{rad},0})}{3\rho_{m,0}^2}\sqrt{\rho_{m,0}a + \rho_{\text{rad},0}} \tag{8.93}$$

which yields

$$t(a) = \frac{2}{3H_0}\sqrt{\frac{a_{r,m}}{\Omega_{m,0}}}\,(a - 2a_{r,m})\sqrt{\frac{a}{a_{r,m}} + 1} \tag{8.94}$$

Now set $a = a_{r,m}$ to find the transition time

$$t_{r,m} = 1.74\times10^{12}s = 5\times10^4\text{yr} \tag{8.95}$$

If $a \ll a_{r,m}$ we are at the radiation-dominated era, for which $a(t) \propto t^{1/2}$, i.e., the expansion was slower. The limit $a \gg a_{r,m}$ corresponds to matter-dominated, for which

$$t(a) = \frac{2}{3}\frac{a^{3/2}}{H_0\sqrt{\Omega_{m,0}}} \tag{8.96}$$

yielding $a(t) \propto t^{2/3}$ as we had already found before. Recasting this in terms of redshift,

$$\frac{t(z)}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}\sqrt{\Omega_{m,0}}} \tag{8.97}$$

At $z = 0$, this yields an age of 12.5 Gyr.

## 8.14 Dark Energy

The Friedmann equation with a cosmological constant is

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho - \frac{1}{3}\Lambda c^2\right]a^2 = -kc^2 \tag{8.98}$$

The extra term can be thought of as potential energy, with

$$U_\Lambda = -\frac{1}{6}\Lambda mc^2 r^2 \tag{8.99}$$

and associated force

$$F_\Lambda = \frac{1}{3}\Lambda mc^2 r\hat{r} \tag{8.100}$$

So that the mechanical energy of a shell is

$$\frac{1}{2}mv^2 - \frac{GM_r m}{r} - \frac{1}{6}\Lambda mc^2 r^2 = -\frac{1}{2}mkc^2\omega^2 \tag{8.101}$$

Notice that the force is directed outward, i.e., it is a centrifugal force, exerting a repulsive force on the shell and accelerating its expansion. The full Friedmann equation with matter (dust), radiation, and dark energy (cosmological constant) is

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad}) - \frac{1}{3}\Lambda c^2\right]a^2 = -kc^2 \tag{8.102}$$

Combining it with the fluid equation yields

$$\frac{d^2a}{dt^2} = \left\{-\frac{4\pi G}{3}\left[\rho_m + \rho_{\rm rad} + \frac{3(P_m + P_{\rm rad})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\}a \tag{8.103}$$

Due to the equivalence of mass and energy we can write a density of dark energy

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \tag{8.104}$$

And we can write more compactly

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda)\right]a^2 = -kc^2 \tag{8.105}$$

Notice that all the factors in the dark energy density are constant. Nothing in it changes in time as the universe expands. So,

$$\rho_\Lambda = \text{const} = \rho_{\Lambda,0} \tag{8.106}$$

As the other densities dilute with the expansion, dark energy eventually dominates.

### 8.14.1 Dark energy pressure

From the fluid equation we can derive the pressure due to dark energy

$$\frac{d}{dt}a^3\rho_\Lambda = -\frac{P_\Lambda}{c^2}\frac{da^3}{dt} \tag{8.107}$$

setting $\rho_\Lambda$ = const and bringing it outside the time derivative

$$P_\Lambda = -\rho_\Lambda c^2 \tag{8.108}$$

i.e., $\omega_\Lambda = -1$. Dark energy exerts negative pressure. The acceleration equation is thus

$$\frac{d^2a}{dt^2} = \left\{-\frac{4\pi G}{3}\left[\rho_m + \rho_{\text{rad}} + \frac{3(P_m + P_{\text{rad}} + P_\Lambda)}{c^2}\right]\right\} a \tag{8.109}$$

with the Friedmann equation in compact form

$$H^2\left[1 - (\Omega_m + \Omega_{\text{rad}} + \Omega_\Lambda)\right] a^2 = -kc^2 \tag{8.110}$$

with

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2}{3H^2} \tag{8.111}$$

If we define $\Omega = \Omega_m + \Omega_{\text{rad}} + \Omega_\Lambda$ then

$$H^2(1 - \Omega)a^2 = -kc^2 \tag{8.112}$$

Measurements observe

$$\Omega_m = 0.27 \tag{8.113}$$
$$\Omega_{\text{rad}} = 8.24 \times 10^{-5} \tag{8.114}$$
$$\Omega_\Lambda = 0.73 \tag{8.115}$$

These numbers are understood in light of the different dilutions laws

$$\rho_m \propto a^{-3} \tag{8.116}$$
$$\rho_{\text{rad}} \propto a^{-4} \tag{8.117}$$
$$\rho_\Lambda \propto a^0 \quad \text{(const)} \tag{8.118}$$

The $\Lambda$-era started at the transition when $\rho_m = \rho_\Lambda$. This occured when

$$a = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = 0.72 \tag{8.119}$$

This corresponds to $z = 0.39$, or 5 billion years ago.

## 8.15 Problems

1. We measure the wavelength of the Lyman-$\alpha$ line ($\lambda_0$ = 1216Å) today, from a source located at $z = 6.0$. Suppose now that we take the measurement again 100 years later. Due to the expansion of the Universe, by how much will the wavelength of the line have shifted? Is this an observable effect? Assume that we live in an $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$ Universe.

2. Derive, from the equations discussed in class, the temperature as a function of time in the era when the Universe was dominated by radiation (assume a flat geometry), and find the time, in seconds after the Big Bang, corresponding to a temperature of 3 MeV.

3. With the James Webb Space Telescope, it should be possible to detect objects as faint as $m_K = 27.4$ in the photometric band $K$ (central $\lambda \approx 2.2\mu m$). At low redshifts, quasars have absolute B-band (central $\lambda \approx$ 4400Å) magnitudes typically of $M_B \approx -23$ and colors $M_B - M_K \approx 2.5$.

   (a) Estimate the highest redshift which these sources can be observed with JWST in the K photometric band.

   (b) Is it realistic to expect JWST to detect, in the K photometric band, these quasars at the redshift you just derived? Hints: examine the assumptions in your estimate, and question if they break down at some redshift. If they do break down, explain why.