1.1 Gravity-Magnetism

- **Define a field**
- **Derive Gauss' law for gravity.**
- **Explain the Divergence Theorem**
- **Apply the Divergence Theorem**
- **Compare and Contrast structure created by gravity and magnetism**
- **Describe qualitatively why flux tubes are buoyant**
- **Explain the existence of shear waves in a magnetic structure**

1.1a What is a field

The study of electric and magnetic fields is too often glanced over, or even not studied at all, at UG physics. When it is studied in depth, it is commonly the most hated, least intuitive, and considered the most useless of the 'core physics' classes. I'd like to change this emphasize from the outset in this course by showing the remarkable properties of the magnetic field. To do so, we first off have to understand what a field is. Here is one good working definition

A field is a model used to explain the influence of one body on another

The type of field that dominates the system, will make that system appear to have a certain type of structure and move in a certain. The type of field that dominates the system will result in certain characteristic size scales and time scales (and thereby velocities). As the certain types of fields create specific types of structure, at certain specific sizes, over certain specific time, (and thereby velocities), you should be able to look at astrophysical object and, just by eye (or maybe with a telescope) work out what is making that object look the way it does.

Newton saw gravity as a force. Einstein showed us it is actually a field, distorting space-time, but we still use Newton's laws to show gravity as a field

$$\mathbf{F} = m\mathbf{g}$$

where g is the gravitational field with a magnitude

$$|\,g\,| = \frac{GM}{r^2} = \frac{|\mathbf{F}|}{m}$$

i.e, has a magnitude given by force per unit mass (N/kg)

But of course, **g** is a vector, so the better (by which I mean, correct) equation to use is

$$\mathbf{g} = -\frac{GM}{|\mathbf{r}|^3}(\mathbf{r})$$

Note that we also have include the fact that gravity is always attractive (negative definite).

Now, let's do one more step, in which we'll show why we should move from this type of 'point source' equation to a much better field equation that deals with extended sources.

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Gauss's Law for Gravity (Note: this derivation will not be examined)

This form of the gravity field equation can be derived from Newton's law, and begins with the basic need to treat mass M as as extended source (i.e., mass is not a point). In other words

$$M = \int \rho(s) d^3\mathbf{s}$$

So let's do that now, so we can all leave Undergraduate physics behind, and move to graduate physics. The Superposition principle states that **g**(**r**), the gravitational field at any point **r**, can be calculated by adding up the contribution to **g**(**r**) due to every bit of mass in the universe. To do this, we integrate over every point **s** in space, adding up the contribution to **g**(**r**) associated with the mass (if any) at **s**, where this contribution is calculated by Newton's law. The result is:

$$\mathbf{g}(\mathbf{r}) = -G \int \rho(s) \frac{(r - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3} d^3\mathbf{s}$$

where ($d^3\mathbf{s}$ stands for $ds_x\, ds_y\, ds_z$, each of which is integrated from -∞ to +∞). Take the divergence of both sides of this equation with respect to **r**, and use the known theorem

$$\nabla . \left(\frac{\mathbf{s}}{|\mathbf{s}|^3} \right) = 4\pi\delta(\mathbf{s})$$

where δ(**s**) is the Dirac delta function and the result is

$$\nabla . \mathbf{g}(\mathbf{r}) = -4\pi G \int \rho(\mathbf{s}) \delta(\mathbf{r} - \mathbf{s}) d^3\mathbf{s}$$

Using the sifting property of the Dirac delta function, we then arrive at the differential form of Gauss's law for gravity

$$\nabla . \mathbf{g}(\mathbf{r}) = -4\pi G \rho(\mathbf{r})$$

We want to compare and contrast gravity and magnetism in order to gain insight into how magnetism governs the structure and dynamics of the heliosphere. As you already know, gravity acts in a straight line (i.e., has no curl), so the complete pair of field equations for gravity are

$$\nabla.\mathbf{g} = -4\pi G\rho \qquad \nabla \times \mathbf{g} = 0$$

Magnetism differs from gravity in an interesting way. For magnetism,

$$\nabla.\mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

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1.1b The Divergence Theorem

The Divergence Theorem is one of the most easy to understand, and yet least understood concepts in nature. To summarize, watch the video again from the 0.1 Why Heliophysics handout.

Take a volume of fluid flow of any substance inside a box. Now consider a number of sources and sinks inside that volume, where a source adds to the substance and a sink removes from the substance. The total flow out of that volume is obtained by adding up the sources and the sinks. Fluid flow is a vector field, and the divergence of the vector defines the difference between a source and a sink - positive divergence is a source, negative divergence is a sink. So if we add up all the sources and sinks inside the volume we can calculate how much fluid is flowing into, or out of, the volume through its boundary. Or, in mathematical language

$$\iiint_v \nabla.\mathbf{v}\, dv = \oiint_s \mathbf{v}.\hat{n}\, da$$

1.1c Applying the Divergence Theorem to the field equations

The divergence theorem allows us to move between point source (integral) and field (differential) form of our equations.

The integral form of Gauss' law for gravity is

$$\oiint_s \mathbf{g}.n\, da = -4\pi GM$$

Apply the divergence theorem

$$\iiint_v \nabla.\mathbf{g}\, dv = -4\pi GM = -4\pi G \iiint_v \rho_m\, dv$$

The both sides of this equation can only be identical at all points in space if the insides of the integrals are equal, and so $\nabla.g = -4\pi G\rho_m$. So, gravity has a conserved source (mass) and gravity causes contraction (negative definite). Gravity forces objects into planes or spheres. Such objects convert gravitational potential energy into kinetic energy in a continual (thermal) manner.

Such objects tend towards order, with long timescales, and low velocities

The integral form of Gauss's law for electrostatics is just Coulamb's law

$$\oiint_s \mathbf{E}.n\, da = Q / \varepsilon_0$$

Apply the divergence theorem

$$\iiint_v \nabla.\mathbf{E}\, dv = Q / \varepsilon_0 = (1/ \varepsilon_0) \iiint_v \rho_c\, dv$$

The both sides of this equation can only be identical at all points in space is if the insides of the integrals are equal, and so $\nabla.\mathbf{E} = \rho_c / \varepsilon_0$ So, electricity has a conserved source (charge). It can be positive or negative but is always conserved. Electrical forces, acting alone, also force objects into ordered structure (the very nature of atoms), with long timescales and slow velocities

The integral forms of Gauss' law for magnetism is that

$$\oiint_s \mathbf{B}.\text{n da} = 0$$

i.e, for every volume in space there is an equal number of field lines entering and exiting the volume. Note, just like Gauss's law for gravity and electrostatics above, this is an experimentally-derived law. It doesn't have to be true (Maxwell's equations don't demand it), but everywhere we look this has been found to be true

Apply the divergence theorem

$$\iiint_v \nabla.\mathbf{B}\ \mathbf{dv} = 0$$

The both sides of this equation can only be identical at all points in space if the inside of the integral is *identically* equal to 0, and so $\nabla.\mathbf{B} = 0$. So magnetism has no unit material source, to put it another way, there is no magnetic monopole. The source of magnetism is in the other field equation.

$$\nabla \times \mathbf{B} = \mu_0\ \mathbf{J}$$

And therein lies the root of the problem - Magnetism has no natural stable solution. Its source is current, which is moving charge, and that isn't stable. It is constantly being generated and dissipated over similar timescales, and so it doesn't reach equilibria. It builds up into an unstable hierarchy and explosively dissipates energy through non-thermal radiation and particle acceleration. It tends to turbulence, with bursty outflows. It leads to a writhing, twisting, pulsing, dynamic, menagerie of structures and events. It results in a pulsating system of loops, sheets, and ropes.

It is primeval.

It is 'Sine Qua Non'

Later in the course we'll come back to these comparisons to derive the two curl equations and discuss the implications of vector potentials. But for now:

First Important fact about magnetism: It has no conserved source

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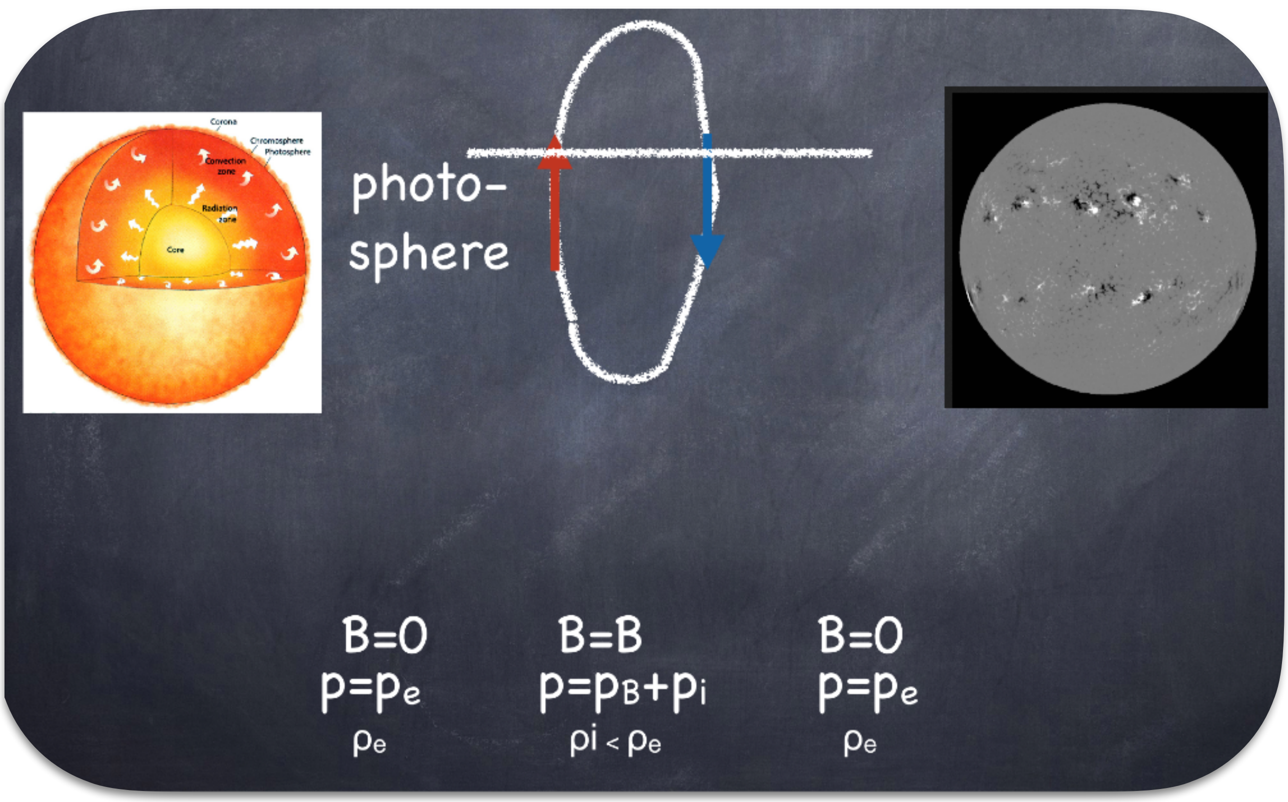
1.1d Buoyancy

Taken together, thermal pressure and gravity often achieve a hydrostatic equilibrium. This balance is the basis for all stars, hence the building block for almost everything else (e.g., planetary systems, galaxies). Magnetism adds in a third force. Later in the course we'll show that magnetism has a positive pressure, therefore expands. This extra magnetic pressure means that regions of strong magnetic field (flux tubes) deep in the convection zone have lower gas pressure than their surrounding non-magnetic regions.

$P_{Gas(Internal)} + P_{magnetic} = P_{Gas(External)}$

Lower gas pressure results in lower density and so these flux tubes rise. As the rise time is roughly the same as the rotation rate of the Sun, this rise results in the symmetry we observe as Hale's Law. Later in the course we'll come back to this concept to study how magnetic flux tubes are stored at the tachocline, transported up through the convection zone, avoid being torn part and emerge in the photosphere. But, for now:

Second Important fact about magnetism: It is buoyant



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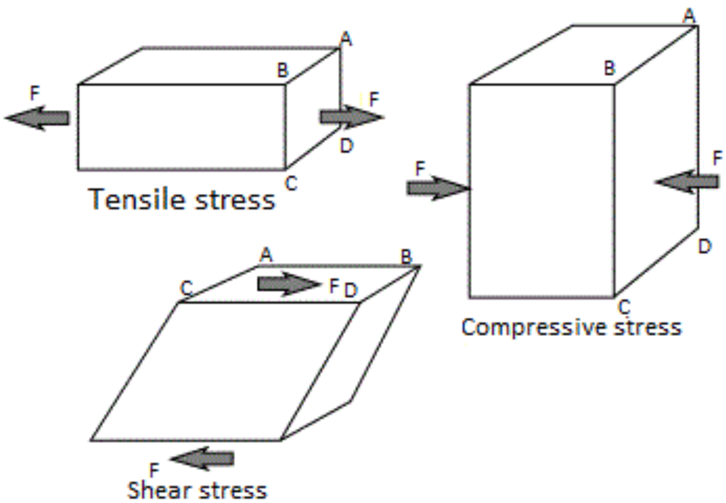
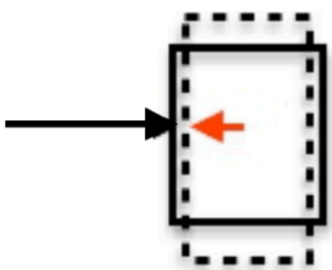
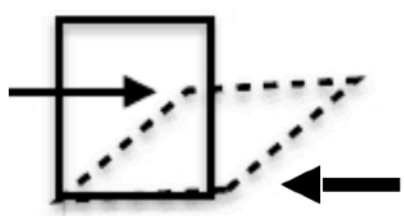
1.1e Shear waves

Shear stress arises from *shear forces*, which are pairs of equal and opposing forces acting on opposite sides of an object. Compressive stress is stress on materials that leads to a smaller volume (opposite to tensile)



Normal fluid resists a compressive stress, but deforms under a shear.





Above left: The (red) restoring force is usually gravity or a pressure gradient. When the restoring force overshoots, we find acoustic waves and gravity waves (both compression waves) at their characteristic timescales and sizescales.

Above right. When a shear is applied, the parcel of gas just deforms. There is no restoring force and a wave will not travel.

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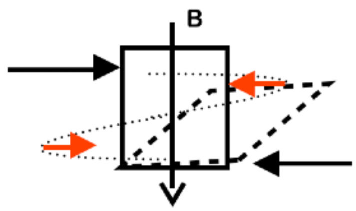
Now consider an electrically conducting fluid with a magnetic field permeating through it. A compressive stress will just push the fluid, bodily, along. But upon shear, now the field will shear with the plasma.



Here we apply the forces (two thick black arrows) perpendicular to the field (arrow pointing down, labelled B). The resulting bending of the field line (dashed line) produces a restoring force (red) that opposes the applied stress.

The overshoot of the restoring force results in a shear wave with its characteristic timescales and size scales, which we'll come back to later in the course. This is entirely unintuitive and Hannes Alfven was laughed at and dismissed for even suggesting it. In 1972, however, he was awarded the Nobel prize for this discovery. This ability to resist stress is key in allowing magnetism to store and release energy and we'll see that this means that magnetism twists matter into shells, tubes and sheets.

Third Important fact about magnetism: It carries shear through tension