1.4 Plasma evolution

- **Describe Faraday's law, Ohm's law, and Ampere's law**
- **Derive the differential form Faraday's law**
- **Derive the generalized of V=IR**
- **Derive the differential form of Ampere's law**
- **Derive and describe the induction equation**

All the objectives below will be introduced after the Mid Term

- **Recognize the form and units of** D_η
- **Compare the diffusive part of the induction equation to the heat transfer equation**
- **Calculate the magnetic Reynolds number in the photosphere**
- **Derive the form of the magnetic Reynolds number**
- **Calculate diffusive size scales and time scales in the solar atmosphere**
- **Describe the granular and super granular flow field velocity, timescales, and size-scales**
- **Show how bright points obtain kG field strength.**
- **Calculate the magnetic pressure to explain why magnetic bright points are bright.**
- **Discuss the horizontal and vertical evolution of flux bundles at the photosphere, and how these combine to make the Sun structured as it is, and evolve as it does.**
- **Show how boring an HD world would be**

The second parameter that we use to describe a plasma is R_m, the magnetic Reynolds number. Where the plasma β describes the structure of the plasma, R_m describes the evolution of the plasma. In working out a path to the Magnetic Reynolds number we will come across the idea of ideal MHD. Then we'll complete this section by showing how the supergranular flow field, with the appropriate Magnetic Reynolds number and plasma β, results in magnetic flux bundles, that appear as congregation of magnetic bright points at the junction of super granular cells.

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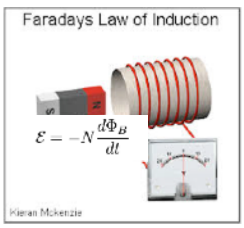
1.4a The induction equation

Let's start with three very important equations and explain these in words. First of all, Faraday's Law of induction (also called the Maxwell-Faraday equation) in c.g.s. units can be express in an differential form as,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{d\mathbf{B}}{dt}$$

The Maxwell-Faraday's equation version of Faraday's law describes how a time varying magnetic field induces a spatially-varying electric field, and vice-verse. This aspect of electromagnetic induction is the operating principle behind many electric generators: for example, a rotating bar magnet creates a changing magnetic field, which in turn generates an electric field in a nearby wire.



Insert the magnet into the coil, and the voltmeter will show that current is flowing. Move the magnet quicker, or have more coils, and the current will be larger. This leads to the integral version of the equation, which states that:

A changing magnetic field causes a force across a conductor

From the integral to the differential: The experiment Faraday described in words led Maxwell to state his result in integral form for the electromotive force, ε, which is c.g.s. units is

$$\epsilon = -\frac{1}{c}N\frac{d\Phi}{dt}$$

In c.g.s., the electromotive force per loop is defined at the force per unit charge, so the Work Done on a unit charge as it moves around a one loop is the Force (per charge) integrated over the distance it moves around one coil.

This can be stated as a loop integral

$$\frac{\epsilon}{N} = \int_l \frac{\mathbf{F}}{q}\,.\,\mathbf{dl} = \int_l \mathbf{E}\,.\,\mathbf{dl} = -\frac{1}{c}\frac{d\Phi}{dt}$$

By using the Kelvin- Stokes theorem[1], the loop integral of any vector is identical to the surface integral of the curl of the vector inside that loop

$$\int_l \mathbf{A}\,.\,\mathbf{dl} = \iint_s \nabla \times \mathbf{A}\,.\,\mathbf{dS}$$, where the loop, l, is around the surface, S

[1] (https://www.khanacademy.org/math/multivariable-calculus/greens-theorem-and-stokes-theorem/stokes-theorem/v/stokes-theorem-intuition)

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So,

$$\int_l \mathbf{E}\,.\,\mathbf{dl} = \iint_s \nabla\times\mathbf{E}\,.\,\mathbf{dS} = -\frac{1}{c}\frac{d\Phi}{dt}$$

And as the flux is also a surface integral

$$\Phi = \iint_s \mathbf{B}\,.\,\mathbf{dS}$$

then

$$\iint_s \nabla\times\mathbf{E}\,.\,\mathbf{dS} = -\frac{1}{c}\frac{d}{dt}\iint_s \mathbf{B}\,.\,\mathbf{dS}$$

And this can only true at all points in space if

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{B}}{dt}$$

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Now, lets look at the resistive form of Ohm's Law

$$\eta \mathbf{J} = \mathbf{E} + (1/c)(\mathbf{v} \times \mathbf{B})$$

The total current density in a system (**J**) is sum of the current caused by two electric fields - the static electric field (**E**) and an induced electric field (**v** X **B**) caused by the motion of the plasma. The η term is a 'resistivity' constant (see box V= IR below). Ohm's law plays the vital role of connecting the magnetic field and the velocity field in one equation. Note they are connected by a 'curl' - the right hand rule - so the induced electric field is maximum when the velocity field and magnetic field are perpendicular to each other, and the induced field is 0 when the velocity field and magnetic field are parallel.

Ohm's law is just a generalization of the standard electric equation V=IR.

Start from the familiar version of Ohm's law, V= IR, where V is the Potential Difference (voltage) across a resistor, R, when some current, I, flows through it. This can be easily described as

The current through a conductor is directly proportional to the voltage difference.

Resistance is proportionally larger for longer wires (large length, l) and smaller cross sectional area, A, so

$$R = \eta l / A$$

where η is just the proportionality constant, known as the resistivity. Ohm's law now becomes

$$V = I\eta \, (l/A)$$

As $\mathbf{J} = I/A$ (current density) and, by definition[2], $\mathbf{E}_{total} = V/l$, then

$$\mathbf{E}_{total} = V/l = I\eta / A = \eta \mathbf{J}$$

Finally, set $\mathbf{E}_{total}$ to be the sum of the static field and the induced field due to motion, and we're recovered the resistive form of Ohm's law as above,

$$\eta \mathbf{J} = \mathbf{E}_{total} = \mathbf{E}_{static} + \mathbf{E}_{motion} = \mathbf{E} + (1/c)(\mathbf{v} \times \mathbf{B})$$

Note, it is common to drop the 'static' from the $\mathbf{E}_{static}$ and so **E** is considered as the static electric field

The equation for the induced field ($\mathbf{E}_{motion}$) comes from the derivation of the Lorentz force, which we will do in the second Module of the course.

[2] potential difference is defined such that 1volt is the potential difference between two plates 1m apart such a force of 1Newtons per Coulamb exists between the plates, i.e., E (N/C) = V/m.

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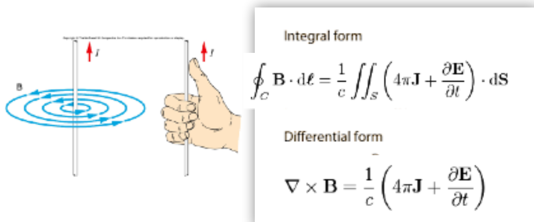
Now, consider Ampere's law (neglecting displacement current dE/dt) is

$$\mathbf{J} = (c/4\pi)\,(\nabla \times \mathbf{B})$$

Simply put, a spatially-varying magnetic field carries a current.

From the integral to the differential:



Commonly understood as
A current-carrying wire will create a 'curled' magnetic field.
Increase the current, or make the displacement electric field change with time, and the magnetic field will be larger.

The experiment Ampere led Maxwell to state a result in more generalized, and integral, for for the magnetic field

$$\int_l \mathbf{B}\,.\,\mathbf{dl} = \frac{1}{c}\iint_s (4\pi\mathbf{J} + \frac{\partial\mathbf{E}}{\partial t})\,.\,d\mathbf{S}$$

Drop the slowly-varying displacement current by $\frac{\partial\mathbf{E}}{\partial t} \longrightarrow 0$, and apply the Kelvin-Stokes theorem

$$\iint_s \nabla\times\mathbf{B}\,.\,\mathbf{dS} = \frac{1}{c}\iint_s (4\pi\mathbf{J})\,.\,d\mathbf{S}$$

And this can only true at all points in space if

$$\nabla\times\mathbf{B} = \frac{1}{c}4\pi\mathbf{J}$$

Note - In 'Ideal MHD' we neglect the displacement current from the more general version of Ampere's law, by assuming that the additional d**E**/dt term is very slow.

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Now, armed with these three equations, we eliminate **E** between these Ohm's Law and Faraday's law, then use Ampere's law to simplify the result.

$$-(\partial \mathbf{B}/\partial t) = c\left(\nabla \times \mathbf{E}\right)$$

$$-(\partial \mathbf{B}/\partial t) = c\left(\nabla \times (\eta\, \mathbf{J} - (1/c)(\mathbf{v} \times \mathbf{B}))\right)$$

$$(\partial \mathbf{B}/\partial t) = \left(\nabla \times (\mathbf{v} \times \mathbf{B})\right) - \left(c\eta\,(\nabla \times \mathbf{J})\right)$$

where we multiplied across the whole equation by -1 and took the 'c' inside the ( ) on the RHS

Having removed **E** from the equation, we can now proceed to use Ampere's law to remove **J**, and leave an equation that just contains the magnetic field and the velocity field.

$$(\partial \mathbf{B}/\partial \mathbf{t}) = \left(\nabla \times (\mathbf{v} \times \mathbf{B})\right) - (c\eta)(c/4\pi)\left(\nabla \times (\nabla \times \mathbf{B})\right)$$

$$= \left(\nabla \times (\mathbf{v} \times \mathbf{B})\right) - (\eta c^2/4\pi)\left(\nabla \times (\nabla \times \mathbf{B})\right)$$

This still looks like a mess, but the key point is that we have now used Ohm's to connect magnetism and fluid velocity, and then eliminated the **E** field and **J** current density completely from the equations.

Now we simplify by using the vector identity

$$\left(\nabla \times (\nabla \times \mathbf{A})\right) = \nabla\,(\nabla . \mathbf{A})) - \nabla^2 \mathbf{A}$$

Such that

$$(\partial \mathbf{B}/\partial t) = \nabla \times (\mathbf{v} \times \mathbf{B}) - D_\eta\,(\,\nabla(\nabla.\mathbf{B}) - \nabla^2 \mathbf{B}\,)$$

We have explicitly assumed that the resistivity is not spatially-varying, and so we took it outside the curl and combined to make a new constant, $D_\eta = \eta c^2/4\pi$, known as electrical diffusivity [3].

As the divergence of the magnetic field is identically 0 (see unit 1.1), then

$$(\partial \mathbf{B}/\partial t) = \nabla \times (\mathbf{v} \times \mathbf{B}) + D_\eta\,(\,\nabla^2 \mathbf{B}\,)$$

which is known as the induction equation

The induction equation states that a local change (in time) of magnetic field is caused by two effects - an advective term (involving **v** x **B**) and a diffusive term (involving $\nabla^2 \mathbf{B}$)

[3] In this course, using c.g.s conventions, we adopt these definitions. However annoyingly, you will find that conventions for this term vary between textbooks. Sometimes D_η is called magnetic diffusivity. Sometimes η is called magnetic diffusivity.