

August 29

- Using the velocity implied from Equation (1.18) as re-expressed in Equation (1.20), we can show that

$$\sigma(E) \equiv \frac{S(E)}{E} \exp(-bE^{-1/2}), \tag{1.24}$$

where

$$b = 31.291 Z_1 Z_2 \mathcal{A}^{1/2} \left[\text{keV}^{1/2} \right]. \tag{1.25}$$

- Now Equation (1.15) becomes

$$\langle \sigma v \rangle = \left(\frac{8}{m\pi} \right)^{1/2} (k_B T)^{-3/2} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/k_B T} dE. \tag{1.26}$$

- Note the two competing effects here: the first exponential increases rapidly with energy, since higher energy nuclei have an easier time to tunnel and this increases the cross section. The second exponential decreases rapidly with energy because of the small probability of there being high energy nuclei. This gives a strongly peaked integrand called the Gamow peak (see sketch).
- The maximum in the curve in energy, known as the “Gamow peak,” is then

$$E_0 = \left(\frac{bk_B T}{2} \right)^{2/3} = 1.22042 (Z_1^2 Z_2^2 \mathcal{A} T_6^2)^{1/3} \text{ [keV]}. \tag{1.27}$$

The notation T_6 is shorthand, in this case, for “millions” of Kelvin. In other words, $T_6 = T \cdot 10^{-6}$, where the real temperature is T .

PROBLEM 1.3: [10 pts]: First, derive the first equality for E_0 in Equation (1.27). Then show that the constant b in Equation (1.25) is correct. Finally, show that the second equality in Equation (1.27) is correct.

PROBLEM 1.4: [5 pts]: Is the energy at the peak of the Gamow curve still consistent with typical nuclei energies in stellar cores? Compute Equation (1.27) for proton-proton collisions at 20 million K, and compare it to the energy of particles in Equation (1.11), both in keV. What does your comparison qualitatively say about the energies of the particles that will participate in reactions with appreciable cross sections?

- Given an $S(E)$, Equation (1.26) can be integrated. A good approximation is to neglect variations of $S(E)$ over the Gamow peak (this doesn’t work when there are resonances). We also approximate the rest of the exponential terms by a gaussian function (see Problem 1.5). One finally gets

$$\langle \sigma v \rangle = \frac{8\sqrt{2}}{9\sqrt{3}} \frac{S(E_0)}{\sqrt{mb}} \eta^2 e^{-\eta}, \tag{1.28}$$

where

$$\eta = \frac{3E_0}{k_B T} = B T_6^{-1/3}, \tag{1.29}$$

and

$$B = 42.487 (Z_1^2 Z_2^2 \mathcal{A})^{1/3}. \tag{1.30}$$

- This equation determines the temperature dependence of the average reaction rate. The rates decrease with η , and so increasing $Z_1 Z_2$ decreases the rates, as well as with \mathcal{A} , since the velocities decrease at fixed energy. And because η varies as $(Z_1 Z_2)^{2/3}$, the temperature sensitivity of the reactions increases quite strongly with nuclear charge. One thing we’ve left out is electron screening which increases reaction rates by cancelling some of the positive charge.

PROBLEM 1.5: [10 pts]: Show that indeed the exponential terms inside the integral in Equation (1.26) can be approximated as a gaussian function in the vicinity of E_0 when assuming a constant $S(E)$. Thus, show that before integration, the integrand can be expressed as

$$\exp\left(-\frac{3E_0}{k_B T}\right) \exp\left[-\frac{(E-E_0)^2}{2\Delta^2}\right], \quad (1.31)$$

where $\Delta = \sqrt{2E_0 k_B T/3}$. (Hint: Taylor expand the *argument* of the exponentials around the Gamow peak). You are not being asked to carry out the integration to derive Equation (1.28).

- Let's carry on to the actual reaction rates by making a simplification. Let's approximate $\langle\sigma v\rangle$ by its value around some T_0 by

$$\langle\sigma v\rangle \approx \langle\sigma v\rangle_0 \left(\frac{T}{T_0}\right)^n, \quad (1.32)$$

where $\langle\sigma v\rangle_0$ is the value at $T = T_0$.

- The temperature dependence of the cross section is now characterized by

$$n = \frac{d \ln \langle\sigma v\rangle}{d \ln T} = \frac{d \ln \langle\sigma v\rangle}{d \ln \eta} \frac{d \ln \eta}{d \ln T} = \frac{\eta - 2}{3}, \quad (1.33)$$

evaluated at $T = T_0$, which is indeed the temperature dependence of the entire energy release.

PROBLEM 1.6: [5 pts]: Show that indeed $n = (\eta - 2)/3$ using Equation (1.33).

- Now we have

$$r_{aA} = \langle\sigma v\rangle_0 n_a n_A \left(\frac{T}{T_0}\right)^n. \quad (1.34)$$

If we re-express the number densities in terms of mass fractions X_i of the nuclei,

$$n_i = \frac{\rho}{m_u} \frac{X_i}{A_i}. \quad (1.35)$$

Then

$$r_{aA} = \langle\sigma v\rangle_0 \frac{X_a X_A}{A_a A_A} \frac{\rho^2}{m_u^2} \left(\frac{T}{T_0}\right)^n. \quad (1.36)$$

We'll come back to this. But note that $\sum_i X_i = 1$.

IN CLASS WORK

Consider 1 hydrogen nucleus and 2 helium nuclei in a 1 cm^3 volume. Using Equation (1.35) compute the number densities of each species, and make sure your answer makes sense.

Answer: The total mass density in this case, with 9 particles, is $\rho = 9 m_u \text{ cm}^{-3}$. We also see that $X_H = 1/9$ and $X_{He} = 8/9$, and $A_H = 1$ and $A_{He} = 4$. For H, we'd have

$$n_H = 9 \frac{1/9}{1} \text{ cm}^{-3} = 1 \text{ cm}^{-3}.$$

For He,

$$n_{He} = 9 \frac{8/9}{4} \text{ cm}^{-3} = 2 \text{ cm}^{-3}.$$

These particle number densities make sense.

1.1.4 Energy release in nuclear reactions

- Just recall some definitions (ignoring electrons):
 - Atomic number Z : number of protons in a nucleus. $Z_{\text{H}} = 1$, $Z_{\text{He}} = 2$, etc. Always an integer.
 - Mass number A : number of protons Z and neutrons N in a nucleus. Always an integer.
 - Atomic mass: The true mass of a single atom (single isotope). A number very nearly equal to the mass number A when expressed in amu. Can be greater or less than A .
 - Atomic weight: The averaged mass over all isotopes of an element (typically what is given in a periodic table). Usually greater than A .
 - Atomic mass unit, amu: $1/12$ the mass of neutral carbon 12.

- Consider the reaction of *nuclei*



Conservation of energy requires

$$E_{a,A} + (m_a + m_A)c^2 = E_{y,Y} + (m_y + m_Y)c^2, \quad (1.38)$$

where the $E_{i,I}$ on each side is the kinetic energy of the center-of-mass of each system. Also included are the rest mass energies of each species.

- We can rewrite Equation (1.38) as

$$E_{y,Y} = E_{a,A} + Q, \quad (1.39)$$

where

$$Q = c^2[m_a + m_A - m_y - m_Y]. \quad (1.40)$$

The quantity Q can be interpreted as the energy released in any reaction, or, the increase in energy for each reaction.

- Note that these reactions are taking place among *nuclei*. However, since charge is conserved, we may replace the nuclear masses implied in the above equations with the atomic masses, because the same number of electron rest masses will be added to both sides of the equation. A small error in the neglected electron binding energy is introduced (of a few eV), but the great convenience in using atomic masses is well worth it.
- Also conserved in these reactions is the number of *nucleons*, and it is convenient to remove their contribution as it will not change the energy budget. The way to do this is to consider that the mass number is the nearest integer to the exact mass of an atom in atomic mass units.
- So consider an atom with Z protons, N neutrons, and $A = Z + N$ nucleons, with atomic mass m . We can derive what's known as the *mass excess* or *mass defect* Δm (which has units of energy):

$$\begin{aligned} \Delta m &= (m - (Z + N)m_{\text{u}})c^2, \\ &= (m - Am_{\text{u}})c^2 \\ &= [m(\text{amu}) - A] c^2 m_{\text{u}}, \end{aligned}$$

or finally,

$$\Delta m = 931.494 \text{ MeV} [m(\text{amu}) - A] \quad [\text{MeV}], \quad (1.41)$$

where $m(\text{amu})$ is the atomic mass of the nucleon in question in amu, and $1m_{\text{u}} = 931.5 \text{ MeV}/c^2$.

- Note, mass excess is really just the difference between the atomic mass of an element (which is usually a number with a very small decimal addition) and its mass number (which is always an integer, A).
- Mass excesses are given in the table in Figure 1.6. As an example of using the above expression to come up with these values, take ${}^4\text{He}$. Its atomic mass $m = 4.002602$. It has 4 nucleons. So

$$\Delta m = (931.494)(4.002602 - 4) = 2.44 \text{ MeV}. \quad (1.42)$$

(slightly different than the table because of updated atomic masses).