Unit 2

Hydrostatics

The goal of this chapter is to provide an overview of general stellar equations and the physical properties to which they correspond.

2.1 Are stars approximately a one-fluid plasma?

- We know stars are mostly ionized. Shouldn't we treat the positive and negative charges as separately?

- If collisions are sufficiently frequent and Maxwellian and have times scales much less than other time scales of interest, then we can use a one-fluid model and generally ignore charge separation.

- The Debye length is the length over which an appreciable electric field can arise. It is roughly a measure of the thermal energy/electric potential energy ratio. If this length is short with respect to the plasma, we can assume charge neutrality.

$$\lambda_D = \sqrt{\frac{\epsilon_0 k T_e}{n_e q_e^2}} \approx 6.9 \left(\frac{T_e}{n_e} \right)^{1/2} \times 10 \text{ m} \tag{2.1}$$

- In the core of the Sun, $n_e = 10^{32}$ m^{-3} and $T_e = 10^7$K, so that $\lambda_D \approx 10^{-11}$ m. In the photosphere, $n_e = 10^{11}$ m^{-3} and $T_e = 5 \times 10^3$ K, so that $\lambda_D \approx 1.5 \times 10^{-3}$ m. For the corona, we may find that $\lambda_D \approx 10$ m.

- So the mean free path of ions (electrons) are much smaller than the scale of the variations of physical quantities. Therefore we can treat most regions of stars as a one-fluid plasma system and use hydrodynamics. In coronae, however, it may be necessary to resort to *plasma physics* where many approximations are no longer valid.

2.2 Time scales of stars

While stars are for the most part static, there are time scales over which change may occur.

2.2.1 Dynamical timescale

- We know that changes happen in stars. Let's consider the smallest timescale t_{dyn}. Start with

$$g = \frac{GM}{R^2}. \tag{2.2}$$

- The time for a particle to fall under gravity is $\ell = 1/2gt^2 \to t = \sqrt{2\ell/g}$. The time scale therefore for a particle to fall, say, a distance $\ell = R/2$ in a star is

$$t_{\mathrm{dyn}} = \left(\frac{R^3}{GM} \right)^{1/2}. \tag{2.3}$$

- In terms of solar values, we find

$$t_{\mathrm{dyn}} \approx 30 \min \left(\frac{R}{R_\odot} \right)^{3/2} \left(\frac{M}{M_\odot} \right)^{-1/2}. \tag{2.4}$$

So since we don't see large-scale changes on such time scales, we know there must be some balance of forces in the Sun.

> **PROBLEM 2.1:** [5 pts]: Derive equation (2.4) by plugging in the constants. Qualitatively, how would the dynamical times scale for a white dwarf star compare to the Sun? A supergiant star?

2.2.2   Thermal timescale

- If a star has no internal energy sources it can generate energy and radiate by contracting. You saw this in Computer Problem 1.1.

- The luminosity L of a star is the energy output per unit time. If we consider that losing its gravitational potential energy is the only real source of energy, then we can calculate the time a star can radiate at a given luminosity. This is the Kelvin-Helmholtz timescale and can be shown to be:

$$t_{\mathrm{KH}} \approx 30 \,\mathrm{Myr} \left(\frac{M}{M_\odot} \right)^2 \left(\frac{R}{R_\odot} \right)^{-1} \left(\frac{L}{L_\odot} \right)^{-1}. \tag{2.5}$$

- In Lord Kelvin's time, this was a problem because we assumed the Sun would be older than this value, since we by then knew the Earth to be at least several billion years old. However, we still didn't know about nuclear energy sources.

> **PROBLEM 2.2:** [5 pts]: Derive equation (2.5).

2.2.3   Nuclear timescale

- We've already studied the fusion of hydrogen into helium, where the energy released can be estimated as $\Delta E = \Delta mc^2$. We know about 0.7% of the mass is lost.

- If this fusion process only takes place in the inner 10% of the Sun, the energy available is about $7 \times 10^{-4} Mc^2$.

- The timescale is

$$\begin{aligned} t_{\mathrm{nuc}} &= 7 \times 10^{-4} \frac{Mc^2}{L} & (2.6) \\ &= 10^{10} \,\mathrm{yr} \left(\frac{M}{M_\odot} \right) \left(\frac{L}{L_\odot} \right)^{-1}. & (2.7) \end{aligned}$$

We know that luminosity is a strong function of mass, so massive stars burn out their energy very quickly.

2.3 Equation of State

2.3.1 Preliminaries

- This section deals with thermodynamics and distribution functions.

- A distribution function simply measures the number density of a species in 6D space of position and momentum.

- If we know this function for a gas, all thermodynamic quantities can be derived (pressure, temperature, density, composition).

- Equations of state relate pressure, temperature, and number of particles.

- An ideal gas is one in which the particles don't interact (except through elastic collisions). They can exchange energy though, but have to conserve it.

- This approximation breaks down when matter is degenerate, and particles begin to "sense" each other and interact in quantum fashion or otherwise.

- An important thermodynamical quantity is the chemical potential μ_c for each species. For classical particles, $\mu_c \to -\infty$, for degenerate fermions $\mu_c \to \epsilon_{\mathrm{F}}$, for bosons $\mu_c = 0$.

- Chemical changes in the gas use the chemical potential to count particle numbers, and thus to achieve a chemical equilibrium (in addition to thermodynamic equilibrium).

- In thermodynamic equilibrium, statistical mechanics tells us the relationship between the number density (of phase space, ie, number per unit volume per unit momentum: $\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}$) of a species

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left\{[E_j + E(p) - \mu_c]/k_{\mathrm{B}}T\right\} \pm 1}, \tag{2.8}$$

where

- j are the possible energy states of the species (like energy levels in an ion), and E_j is the energy of that level
- $E(p)$ is the kinetic energy
- g_j is the degeneracy of state j (number of states with same energy)
- $\pm$ is either for fermions or bosons, respectively.

We will come back to this frequently.

- To find the number density (particles cm^{-3}) we integrate $n(p)\,\mathrm{d}^3p$ over momentum space (assumed to be symmetric)

$$n = \int_p 4\pi p^2 n(p)\,\mathrm{d}p. \tag{2.9}$$

- To remain completely general, the kinetic energy of a particle of rest mass m is

$$E(p) = (p^2c^2 + m^2c^4)^{1/2} - mc^2. \tag{2.10}$$

IN CLASS WORK

What does this expression reduce to in the nonrelativistic limit?

Answer: In this limit, we note that $pc \ll mc^2$, so one can expand the term in the square root:

$$
\begin{aligned}
E(p) &= mc^2 \left(1 + \frac{p^2 c^2}{m^2 c^4} \right)^{1/2} - mc^2, \\
&\approx mc^2 \left(1 + \frac{1}{2} \frac{p^2 c^2}{m^2 c^4} \right) - mc^2, \\
&\approx \frac{p^2}{2m},
\end{aligned}
$$

which is the expression we'd expect.

- Now we can define three general quantities:

    1. Velocity:
    $$
    v = \frac{\partial E}{\partial p}. \tag{2.11}
    $$

    2. Pressure:
    $$
    P = \int_{\boldsymbol{p}} n(\boldsymbol{p}) \boldsymbol{v} \cdot \boldsymbol{p} \, \mathrm{d}^3 \boldsymbol{p} = \frac{1}{3} \int_p n(p) v \, p \, 4\pi p^2 \, \mathrm{d}p, \tag{2.12}
    $$

    where the last equality comes from assuming isotropy of pressure.

    3. Internal energy density (energy per unit volume):
    $$
    u = \int_p n(p) E(p) 4\pi p^2 \, \mathrm{d}p. \tag{2.13}
    $$

- These general considerations can soon be applied to specific cases.