September 12 . . . .

- One cleaner way of writing this is

$$\mu^{-1} = \frac{\sum_i X_i/A_i(1+Z_i)}{\sum_i X_i} = \sum_i \frac{X_i}{A_i}(1+Z_i). \tag{2.20}$$

- For example, for a neutral gas we have

$$\mu = \frac{\sum_i X_i}{\sum_i X_i/A_i} = \left(\sum_i \frac{X_i}{A_i}\right)^{-1} \approx \left(X + \frac{Y}{4} + \frac{Z}{\overline{A}_i}\right)^{-1}, \tag{2.21}$$

where it is standard to write mass fractions X for hydrogen, Y for helium, and Z for everything else (metals), where $X + Y + Z = 1$.

- $\overline{A}_i$ is an average over metals, which at solar composition is about 15.5.

- For a fully ionized gas

$$\mu^{-1} \approx \sum_i \frac{X_i}{A_i}(1+Z_i) \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z, \tag{2.22}$$

or

$$\mu \approx \frac{4}{3+5X-Z}, \tag{2.23}$$

where for metals we usually approximate $(Z_i+1)/A_i \approx 1/2$ (roughly equal number of protons and neutrons). We've eliminated Y in this expression through $Y = 1 - X - Z$.

IN CLASS WORK

Compute the mean molecular weight for (1) the ionized solar photosphere, where we have 90% hydrogen, 9% helium, and 1% heavy elements, (2) the ionized solar interior where 71% hydrogen, 27% helium, and 2% heavy elements, (3) completely ionized hydrogen, (4) completely ionized helium, and finally (5) neutral gas at the solar interior abundance.

Answer: (1) For the photosphere we can write

$$\mu^{-1} = 0.9\frac{2}{1} + 0.09\frac{3}{4} + 0.01\frac{1}{2} = 1.8725,$$

or $\mu \approx 0.53$.
(2) For the interior we can write

$$\mu^{-1} = 0.71\frac{2}{1} + 0.27\frac{3}{4} + 0.02\frac{1}{2} = 1.63,$$

or $\mu \approx 0.61$.
(3) For hydrogen, we will take $X = Z = A = 1$, and find then that $1/\mu = 2$.
(4) For helium, $X = Z = 0$ and $Y = 1$, so $\mu = 4/3$.
(5) For a neutral gas, we have

$$\mu^{-1} = 0.71 + 0.27\frac{1}{4} + 0.02\frac{1}{15.5} = 0.779,$$

or $\mu \approx 1.28$.

- From the above, we can also consider separately the mean molecular weight for ions and electrons.

- For ions, define $\mu_{\rm I}$ as
$$n_{\rm I} = \frac{\rho}{\mu_{\rm I} m_{\rm u}}. \tag{2.24}$$
Recall that
$$n_{\rm I} = \sum_j n_{j,{\rm I}} = \frac{\rho}{m_{\rm u}} \sum_j \frac{X_j}{A_j}. \tag{2.25}$$
So that
$$\mu_{\rm I} = \left(\sum_j \frac{X_j}{A_j} \right)^{-1}. \tag{2.26}$$
- This result should make sense, since above in Equation (2.21) we did not consider electrons.
- For electrons it's a bit harder since not all electrons need be free. But we will still define the *mean molecular weight per election* $\mu_{\rm e}$:
$$n_{\rm e} = \frac{\rho}{\mu_{\rm e} m_{\rm u}} \tag{2.27}$$
- Fully ionized, each atom contributes Z electrons. If an ion is partially ionized, we can consider the fraction yZ. (To compute the proper fraction of ionization of a gas (n_e), one needs to use the *Saha equation*).
- As before then
$$n_e = \sum_j n_{e,j} = \sum_j n_{j,{\rm I}} y_j Z_j = \frac{\rho}{m_{\rm u}} \sum_j \left(\frac{X_j}{A_j} \right) y_j Z_j, \tag{2.28}$$
which defines
$$\mu_e = \left(\sum_j \frac{X_j y_j Z_j}{A_j} \right)^{-1}. \tag{2.29}$$
- So finally
$$n = n_e + n_{\rm I} = \frac{\rho}{\mu m_{\rm u}}, \tag{2.30}$$
where
$$\mu = \left(\frac{1}{\mu_{\rm I}} + \frac{1}{\mu_e} \right)^{-1}. \tag{2.31}$$

IN CLASS WORK

Compute an expression for $\mu_{\rm e}$ in the deep stellar interior as a function only of X. Ignore metals.

Answer: Fully ionized case. We can write

$$\begin{aligned} \mu_{\rm e} &\approx \left(\frac{1}{1} X + \frac{2}{4} Y \right)^{-1} \\ &= \left(X + \frac{1}{2}(1 - X) \right)^{-1} \\ &= \left(\frac{X+1}{2} \right)^{-1} = \frac{2}{1+X}. \end{aligned}$$

This should make sense. For a full H gas, the mean mass of particles per number of electrons (1/1) is 1. For a He gas ($X = 0$), we have a mass of 4 divided by 2 electrons, or $\mu_e = 2$.

2.3.3 Maxwell-Boltzmann statistics

- The relation of Equation (2.8) to classical probability functions is found through Equation (2.36) and Equation (2.37).
- In momentum space
$$f(p)\,\mathrm{d}p = \frac{4\pi}{(2\pi m k_\mathrm{B} T)^{3/2}} e^{-p^2/2mkT} p^2\,\mathrm{d}p. \tag{2.32}$$
- In energy space
$$f(E)\,\mathrm{d}E = \frac{2}{\sqrt{\pi}(k_\mathrm{B} T)^{3/2}} e^{-E/kT} \sqrt{E}\,\mathrm{d}E. \tag{2.33}$$
- In velocity space
$$f(v)\,\mathrm{d}v = 4\pi \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} e^{-mv^2/2kT} v^2\,\mathrm{d}v. \tag{2.34}$$

2.3.4 Ideal monatomic gas

- As a first demonstration, we consider a gas of single species nonrelativistic particles. We will be using Equation (2.8).
- Their energy is $E = p^2/2m$. Consider one energy level $E_j = E_0$.
- For this system, the chemical potential goes to negative infinity (as we'll see), so the exponential term is large, and the ± 1 term can be safely ignored.
- The number density of particles in any given momentum state p is
$$n(p) = \frac{g}{h^3} e^{-p^2/2mkT} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}, \tag{2.35}$$
and so the total number density for all momenta is
$$n = \frac{4\pi g}{h^3} \int_0^\infty p^2 e^{-p^2/2mkT} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}\,\mathrm{d}p. \tag{2.36}$$
- The integral is straightforward and gives an expression
$$n = \frac{(2\pi m k_\mathrm{B} T)^{3/2} g}{h^3} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}. \tag{2.37}$$
- Another way to write this is
$$e^{\mu_\mathrm{c}/kT} = \frac{nh^3}{g(2\pi m k_\mathrm{B} T)^{3/2}} e^{E_0/kT}. \tag{2.38}$$
Since we are assuming that the term on the left is small (since $\mu_\mathrm{c} \ll -1$), then the right hand side must also be small. Specifically, $nT^{-3/2}$ cannot be too large. If that were the case, then we would not be able to ignore the ± 1 term in the distribution function.
- Returning to the definition of gas pressure in Equation (2.12), we can compute the integral to find
$$P = g \frac{4\pi}{h^3} \frac{\pi^{1/2}}{8m} (2mk_\mathrm{B} T)^{5/2} e^{-E_0/kT} e^{\mu_\mathrm{c}/kT}. \tag{2.39}$$
- Using the generalized number density from Equation (2.37), this gives what you thought it would
$$P = nk_\mathrm{B} T\ [\mathrm{dyne\,cm^{-2}}]. \tag{2.40}$$
This is the equation of state for an ideal gas.

- Similarly we can compute the internal energy density from Equation (2.13)

$$u = \frac{3}{2} n k_{\rm B} T = \frac{3}{2} P; \; [\mathrm{erg\, cm^{-3}}]. \tag{2.41}$$

- Note that the units of pressure and internal energy density are the same.
- From what we saw before with the mean molecular weight, we can also express these quantities as

$$n = \frac{\rho}{\mu m_{\rm u}}, \tag{2.42}$$

$$P = \frac{\rho k_{\rm B} T}{\mu m_{\rm u}}, \tag{2.43}$$

$$P = \frac{\rho R T}{\mu}, \tag{2.44}$$

where μ is the mean molecular weight, and $R = k_{\rm B}/m_{\rm u}$ is the ideal gas constant $R = 8.31 \times 10^7\,\mathrm{erg\,K^{-1}\,mol^{-1}}$.

EXAMPLE PROBLEM 2.1: Instead of arriving at Equation (2.41) through the energy formulation, one can use velocity to show that the average internal kinetic energy per particle is $3/2 k_{\rm B} T$. Hint: Start with the Maxwellian distribution for a classical gas in velocity (Equation (2.34)) and then compute the mean square speed $\langle v^2 \rangle$. The integration limits of v should be from zero to infinity.

Answer: The mean square speed can be written as

$$\langle v^2 \rangle = \int_0^\infty v^2 f(v) \mathrm{d}v,$$

where $f(v)$ is the Maxwell distribution. One can (and should) use a table of integrals to find that

$$\int_0^\infty x^n e^{-ax^2} \mathrm{d}x = \frac{(2k-1)!!}{2^{k+1} a^k} \left(\frac{\pi}{a}\right)^{1/2},$$

where, in our case, $n = 2k$ and $a = m/2k_{\rm B}T > 0$. Note the double factorial, which, for $k = 2$, is $3 \times 1 = 3$. The result is

$$\langle v^2 \rangle = 4\pi \left(\frac{m}{2\pi k_{\rm B} T}\right)^{3/2} \cdot \frac{3}{8} \frac{\pi^{1/2}}{a^{5/2}},$$

and after cancellation becomes

$$\langle v^2 \rangle = \frac{3 k_{\rm B} T}{m}.$$

The kinetic energy is then $1/2 m \langle v^2 \rangle = 3/2 k_{\rm B} T$, precisely what we set out to prove.