**September 19** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

### 2.3.7 Radiation pressure

- Particles are not the only source of pressure in a star. The radiation field of photons can also exert a pressure.
- We already have an expression for the general pressure in Equation (2.12).
- With a degeneracy factor $g = 2$ (photons have 2 spin states, or polarizations, each with the same energy at fixed frequency), chemical potential (bosons) $\mu_c = 0$, $E = pc$, and $E_j = 0$, the distribution function Equation (2.8) is

$$n(p) = \frac{2}{h^3} \frac{1}{\exp\left(pc/kT\right) - 1}. \tag{2.58}$$

- From Equation (2.12), we thus have

$$P_{\rm rad} = \frac{8\pi c}{3h^3} \int_0^\infty \frac{p^3}{{\rm e}^{pc/kT} - 1}\, {\rm d}p, \tag{2.59}$$

  where we made the substitution $v \equiv c$ for photons.

- The integral can be solved (Problem 2.4) to give

$$P_{\rm rad} = \frac{1}{3} a\, T^4, \tag{2.60}$$

  where $a = 4\sigma/c = 7.5 \times 10^{-15}\,{\rm erg\,cm^{-3}\,K^{-4}}$.

- Similarly as before, the energy density

$$u_{\rm rad} = aT^4 = 3P_{\rm rad}. \tag{2.61}$$

**PROBLEM 2.4:** [10 pts]: Carry out the integral in Equation (2.59) to show that

$$a = \frac{8\pi^5}{15} \frac{k^4}{h^3 c^3}. \tag{2.62}$$

Hint: make the substitution $x = pc/kT$.

### 2.3.8 Density-temperature equation of state landscape

- Putting the previous sections together, the pressure of stellar matter through the equation of state in general is

$$P = P_{\rm ion} + P_{\rm e} + P_{\rm rad}. \tag{2.63}$$

- In some cases, the electron pressure is from degenerate particles. In rare cases, the ions can become degenerate too.
- Not all of these pressure terms contribute equally to the total pressure at any given time, as you saw in Problem 2.3.
- Consider the total gas pressure of an ideal gas as

$$P_{\rm gas}^{\rm ideal} = P_{\rm ion} + P_{\rm e} = \frac{\rho k_{\rm B} T}{\mu m_{\rm u}}. \tag{2.64}$$

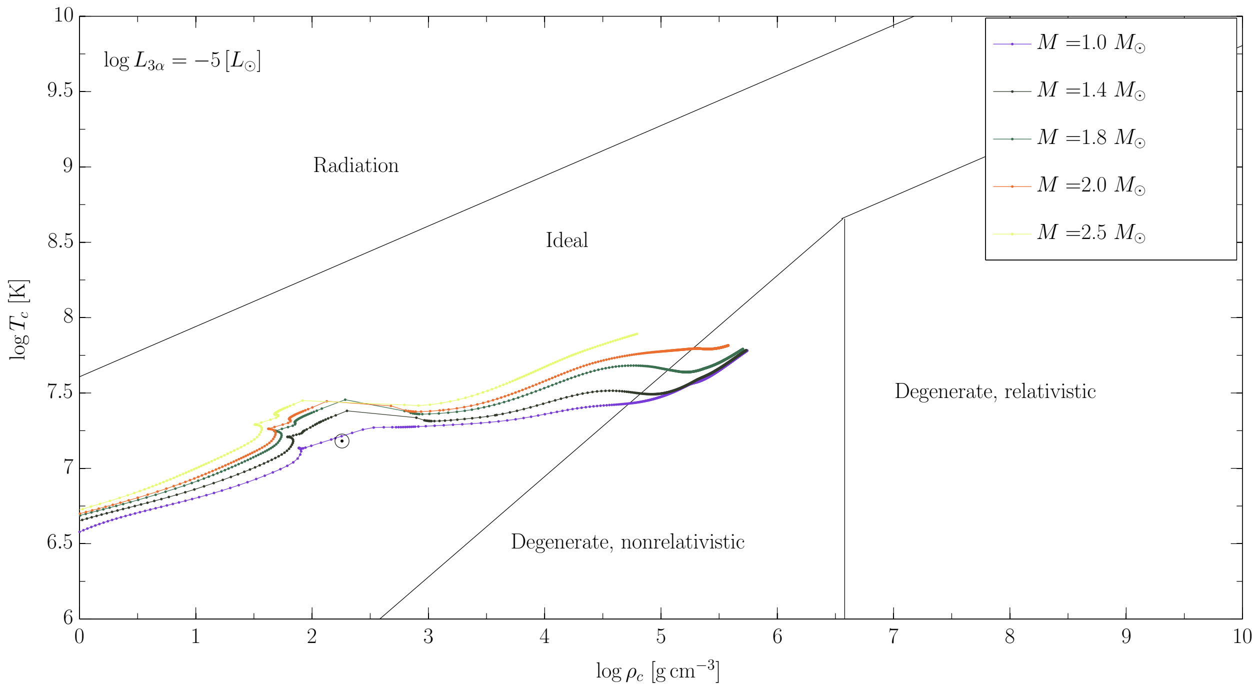


**Figure 2.1:** Stellar matter under core conditions. These boundary regimes are computed using $\mu_{\rm e} = 2$ and $\mu = 0.5$. A few example model tracks (central values as a funciton of time) are shown for different masses, until the point when the contribution to the luminosity by the triple$-\alpha$ process is 1 part in $10^5$ solar luminosities, as denoted in the figure. In practice, these stars are near their respective tip of the red-giant branch. The current location of the Sun is given by its symbol.

- It is useful to compare regions where this and $P_{\rm e}^{\rm deg}$ and $P_{\rm rad}$ compete and transition to one another.
- First consider where an ideal gas transitions to a degenerate nonrelativistic one. Equating Equation (2.53) and Equation (2.64) gives

$$\frac{\rho}{\mu_{\rm e}^{5/2}} = \left(\frac{k_{\rm B}}{C\mu m_{\rm u}}\right)^{3/2} T^{3/2}, \tag{2.65}$$

  where $C$ is the constant prefactor in Equation (2.53) .
- For large densities or low temperatures, i.e., when $\rho T^{-3/2} >$ const, the gas is dominated by degenerate pressure. This is shown by the line of a slope of 2/3 in Figure 2.1.
- When electron speeds become relativistic, Equation (2.56) becomes appropriate, and when equated with the ideal gas pressure yields

$$\frac{\rho}{\mu_{\rm e}^{4}} = \left(\frac{k_{\rm B}}{D\mu m_{\rm u}}\right)^{3} T^{3}, \tag{2.66}$$

  where $D$ is the constant prefactor in Equation (2.56).
- This is shown in Figure 2.1 with the line of slope 1/3 at high temperature and density.
- In the degenerate regime, the transition from non- to relativistic is found by equating Equation (2.53) and Equation (2.56), which is independent of temperature (since these are completely degenerate systems):

$$\frac{\rho}{\mu_{\rm e}} = \left(\frac{D}{C}\right)^{3}. \tag{2.67}$$

  This is given in the figure by the vertical line.

- Finally, we can determine where radiation pressure starts to exceed ideal gas pressure. We use Equation (2.60) to find
$$\frac{\rho}{\mu} = \frac{1}{3}\frac{am_{\mathrm{u}}}{k_{\mathrm{B}}}T^3. \tag{2.68}$$
- In Figure 2.1 this is shown by the line at the bottom right of slope 1/3.

### 2.3.9 Thermodynamics of an ideal gas

- Here we consider quasistatic changes to the state of a nondegenerate gas.
- As already stated, the internal energy per unit volume of an ideal gas is
$$u = \frac{3}{2}nk_{\mathrm{B}}T = \frac{3}{2}\frac{\rho k_{\mathrm{B}}T}{\mu m_p} = \frac{3}{2}P. \tag{2.69}$$
- Therefore the average energy per particle is $3/2k_{\mathrm{B}}T$. Example problem 2.1 arrived at this in a slightly different way.
- We define the specific volume $V$ as the volume corresponding to unit mass, $V = \text{volume/mass} = 1/\rho$. The specific internal energy is the internal energy per unit mass $U = u/\rho$:
$$U = \frac{3}{2}\frac{k_{\mathrm{B}}T}{\mu m_u}. \tag{2.70}$$
- Remember that the first law of thermodynamics tells us that we can (slowly) change the internal energy of gas by adding heat or doing work:
$$\mathrm{d}U = \mathrm{d}Q + \mathrm{d}W, \tag{2.71}$$
where $U$ is the specific internal energy of the matter, $V$ is the specific volume it occupies, and $\mathrm{d}Q$ is some amount of heat added to it.
- The work done is to contract or expand it, so $\mathrm{d}W = -P\mathrm{d}V$.
- The more proper form for our use is
$$\mathrm{d}Q = \mathrm{d}U + P\mathrm{d}V. \tag{2.72}$$
This heat partly changes the internal energy of the matter and also potentially changes the volume.
- Keeping the volume constant the first law of thermodynamics becomes
$$c_V = \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_V = \frac{\mathrm{d}U}{\mathrm{d}T} = \frac{3}{2}\frac{k_B}{\mu m_u} = \frac{3}{2}\frac{R}{\mu}. \tag{2.73}$$
This is the specific heat at constant volume.
- Now rewrite the first law
$$\mathrm{d}Q = \mathrm{d}U + P\mathrm{d}V + V\mathrm{d}P - V\mathrm{d}P = \mathrm{d}U - V\mathrm{d}P + \mathrm{d}(PV), \tag{2.74}$$
$$\mathrm{d}(PV) = \mathrm{d}(Nk_{\mathrm{B}}T) = \frac{k_{\mathrm{B}}}{\mu m_u}\mathrm{d}T, \tag{2.75}$$
remembering that $N = nV = n/\rho = \rho/(\mu m_u \rho) = 1/\mu m_u$.
- Then
$$\mathrm{d}Q = \mathrm{d}U - V\mathrm{d}P + \frac{k_{\mathrm{B}}}{\mu m_p}\mathrm{d}T = \frac{5}{2}\frac{k_{\mathrm{B}}}{\mu m_u}\mathrm{d}T - V\mathrm{d}P, \tag{2.76}$$
by using Equation (2.70).

- Therefore the specific heat at constant pressure is

$$c_P = \frac{5}{2}\frac{k_{\rm B}}{\mu m_u} = \frac{5}{2}\frac{R}{\mu}.. \tag{2.77}$$

  Note that $c_P - c_V = R/\mu$.

- Note also the ratio of specific heats, $\gamma = c_P/c_V$, which for an ideal gas $\gamma = 5/3$ since the specific heats are constants.

- An **adiabatic process** is one in which no heat is added to the gas ($\mathrm{d}Q = 0$).

- In this special case, we can find expressions relating changes in $P$ and $V$. Using the above expressions we can show

$$c_V\left(\frac{\mathrm{d}P}{P} + \frac{\mathrm{d}V}{V}\right) = -\frac{k_{\rm B}}{\mu m_p}\frac{\mathrm{d}V}{V} = (c_V - c_P)\frac{\mathrm{d}V}{V}. \tag{2.78}$$

- Finally we see that

$$\frac{\mathrm{d}P}{P} = -\gamma\frac{\mathrm{d}V}{V} = \gamma\frac{\mathrm{d}\rho}{\rho} = -\frac{\gamma}{1-\gamma}\frac{\mathrm{d}T}{T}. \tag{2.79}$$

- Since $\gamma$ is constant in this case, such equations can be readily integrated to yield relations such as $PV^\gamma = \mathrm{const}$.

- Using the ideal gas law we can also write (just in terms of $P$, $T$, and $V$):

$$\left(\frac{\partial \ln P}{\partial \ln V}\right)_s = -\gamma \equiv -\Gamma_1 \tag{2.80}$$

$$\left(\frac{\partial \ln P}{\partial \ln T}\right)_s = \frac{\gamma}{\gamma - 1} \equiv \frac{\Gamma_2}{\Gamma_2 - 1} \tag{2.81}$$

$$\left(\frac{\partial \ln T}{\partial \ln V}\right)_s = 1 - \gamma \equiv 1 - \Gamma_3. \tag{2.82}$$

- The $s$ means adiabatic, or at constant entropy, where $\mathrm{d}Q = T\mathrm{d}S$.

**PROBLEM 2.5:** [5 pts]: Derive the 2 equations (2.78) and (2.79).