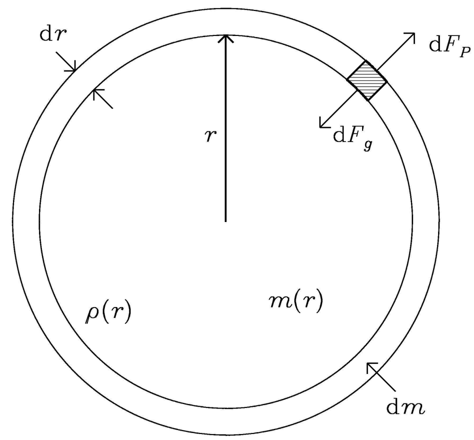

Figure 2.6: Schematic for deriving hydrostatic equilibrium. From ?.

<u>**September** 26</u> ....

2.4 Hydrostatic Equilibrium

2.4.1 Derivation

- Consider Figure 2.6. Take a thin mass element in a star of thickness $\mathrm{d}r$ and surface $\mathrm{d}A$ at radius r (and thus of mass $\mathrm{d}m = \rho \mathrm{d}r \mathrm{d}A$) from the center.
- The gravitational force on that mass element is

$$\mathrm{d}F_g = -\frac{G\left[\rho(r)\mathrm{d}r\mathrm{d}A\right]m(r)}{r^2}, \tag{2.104}$$

  directed radially inward.
- In equilibrium, this force is balanced by an outward pressure force acting at r and $r+\mathrm{d}r$ ($P = \mathrm{d}F/\mathrm{d}A$)

$$\mathrm{d}F_P = \left[P(r) - P(r+\mathrm{d}r)\right]\mathrm{d}A = -\frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r\mathrm{d}A, \tag{2.105}$$

  by using the definition of the derivative.
- In hydrostatic equilibirum, $\mathrm{d}F_g + \mathrm{d}F_P = 0$, and so

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho(r)m(r)}{r^2}. \tag{2.106}$$

- We will commonly see this written in vector form as

$$\boldsymbol{\nabla}_{\boldsymbol{r}} P = \rho \boldsymbol{g}. \tag{2.107}$$

- Note that the mass element in the thin shell can be expressed as

$$\mathrm{d}m = 4\pi r^2 \rho \mathrm{d}r, \tag{2.108}$$

  or

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho r^2. \tag{2.109}$$

- Thus the mass as a function of radius is found by

$$m(r) = \int_0^r 4\pi r'^2 \rho(r') \mathrm{d}r'. \tag{2.110}$$

IN CLASS WORK

Estimate the central pressure of the Sun from the equation of hydrostatic equilibrium. Compare to the tabulated values. Try to put all final expressions in terms of scaled solar values as we have been doing.

Answer: The simplest thing one can do is ignore the derivatives in the equilibrium expression and assume the density is constant:

$$\frac{\partial P}{\partial r} = -\frac{G\rho(r)M(r)}{r^2} \tag{2.111}$$

$$\frac{P_c}{R} = \frac{3}{4\pi}\frac{GM^2}{R^5} \tag{2.112}$$

$$P_c = \frac{3}{4\pi}\frac{GM_\odot^2}{R_\odot^4}, \tag{2.113}$$

where we've replaced all values by the gross solar ones. For a general star we can show

$$P_c \approx 2.69 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \mathrm{dyne\,cm^{-2}}.$$

We know that 1 dyne cm^{-2} $= 0.1$ N m^{-2} [Pa], so we are only off in pressure by about 2 orders of magnitude when compared to the tabulated value of 2.3×10^{17} dyne cm^{-2}. Not so bad actually. The tabulated value is roughly

$$P_c \approx \frac{261}{4\pi}\frac{GM_\odot^2}{R_\odot^4} \quad \text{(tabulated)}$$

PROBLEM 2.6: [10 pts]: (a) Do the same type of calculation as for the central pressure to find the central temperature and compare to the table value. Use the same mass fractions as in Problem 2.3. (b) Then, using the expressions you now have for the gas pressure and temperature, show that we can ignore the radiation pressure for the Sun in the core. I.e., show that $P_{\mathrm{R}}/P_{\mathrm{G}} \approx c(M/M_\odot)^2$, where c is a smallish number. Try to put all final expressions in terms of scaled solar values as we have been doing.

PROBLEM 2.7: [5 pts]: In Equation (2.113) we found a cheap and dirty estimate of the central pressure. Now, using Equations (2.106) and (2.109), find a **lower bound** for the pressure at the center of the Sun. First find an expression for $\mathrm{d}P/\mathrm{d}m$ and integrate from core to surface. Making a simple assumption in the integrand allows you to argue this is really a lower limit. (You may want to read Problem 2.8 before you try this one). Compare again to the previous result of the in class problem by expressing your final answer in terms of

$$P_c = \mathrm{const} \times \frac{GM_\odot^2}{R_\odot^4}, \tag{2.114}$$

where the constant is really the key quantity in your computation.

PROBLEM 2.8: [5 pts]: Let's improve the lower limit now (i.e., make it a bit larger). All that is needed is to assume a mean density that is a decreasing function of r such as

$$\overline{\rho(r)} = \frac{m}{4\pi r^3/3}.$$

Use this at the right step in Problem 2.7 to get a new lower limit, again expressed as

$$P_c = \text{const} \times \frac{GM_\odot^2}{R_\odot^4}.$$

2.5 The Virial Theorem

- The Sun is in hydrostatic equilibrium. It is not necessarily in thermal equilibrium, but let's see.
- First a quick look at numbers. Consider the **thermal** energy (internal energy density) of an ideal monatomic gas

$$u = \frac{3}{2}\frac{\rho k_\mathrm{B} T}{\mu m_\mathrm{u}}. \tag{2.115}$$

- Integrated over the star $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ and looking at the specific internal energy $\mathrm{d}U = u\mathrm{d}V$, we find

$$U = \int_0^R \left(\frac{3}{2}\frac{k_\mathrm{B}}{\mu m_\mathrm{u}} T\right) \rho 4\pi r^2 \mathrm{d}r = \int_0^M \left(\frac{3}{2}\frac{k_\mathrm{B}}{\mu m_\mathrm{u}} T\right) \mathrm{d}m = \frac{3}{2}\frac{k_\mathrm{B}}{\overline{\mu} m_\mathrm{u}} \overline{T} M, \tag{2.116}$$

  where the overbars denote average values (we ignore radial dependence of temp. and composition, for now). This is dirty.

- Now consider the gravitational potential energy

$$E_\mathrm{G} = -\int_0^M \left(\frac{Gm}{r}\right) \mathrm{d}m = -\frac{GM^2}{R}. \tag{2.117}$$

- Using some typical solar values values: $k_\mathrm{B} = 1.38 \times 10^{-16} \mathrm{erg\,K^{-1}}$, $\overline{\mu} = 0.62$, $m_\mathrm{u} = 1.66 \times 10^{-24}$g, mean temperature $\overline{T} = 10^7$K, $M_\odot = 1.99 \times 10^{33}$g, $G = 6.67 \times 10^{-8} \mathrm{dyne\,cm^2\,g^{-2}}$, $R_\odot = 6.96 \times 10^{10}$cm. We find that

$$U \approx +4.0 \times 10^{48}\,\mathrm{erg}, \tag{2.118}$$
$$E_G \approx -3.8 \times 10^{48}\,\mathrm{erg}. \tag{2.119}$$

- Why are these two numbers so close, even the same order of magnitude? This suggests something deeper is happening.