October 17 . . . .

3.1.6 Eddington Luminosity

- Let's take a quick stop and look at an interesting consequence of diffusive radiation.
- Consider the near-surface of a star where radiation into space dominates over the gas pressure
- We found before that

$$P_{\rm rad} = \frac{1}{3} a T^4,$$

so

$$\frac{{\rm d}P_{\rm rad}}{{\rm d}T} = \frac{4}{3} a T^3 \frac{{\rm d}T}{{\rm d}r}. \tag{3.26}$$

- The radiative flux can then be expressed as (see Equation (3.7))

$$F_{\rm rad} = -\frac{c}{\kappa_{\rm R}\rho} \frac{{\rm d}P_{\rm rad}}{{\rm d}r}. \tag{3.27}$$

- Let's assume that it might be possible that the radiation pressure overcomes gravity.
- In that case consider hydrodstatic (non)equilibrium to occur when

$$-\frac{{\rm d}P_{\rm rad}}{{\rm d}r} > \rho g. \tag{3.28}$$

- Using Equation (3.27) this becomes

$$\frac{F_{\rm rad}\kappa_{\rm R}}{c} > g. \tag{3.29}$$

- If we just consider the area near the surface and look and consider $L = 4\pi R^2 F_{\rm rad}$, then

$$\frac{\kappa_{\rm R} L}{4\pi c G M} > 1. \tag{3.30}$$

- We can write the quantity on the left as

$$\frac{\kappa_{\rm R} L}{4\pi c G M} = 7.8 \times 10^{-5} \kappa_{\rm R} \left(\frac{L}{L_\odot}\right) \left(\frac{M}{M_\odot}\right)^{-1}, \tag{3.31}$$

which shows that this number is typically small, much less than 1.

- For massive stars, however, the luminosities can get quite large.
- So if we define the Eddington luminosity at the point where this number is unity, then

$$\frac{L_{\rm Edd}}{L_\odot} \approx 3.7 \times 10^4 \left(\frac{M}{M_\odot}\right), \tag{3.32}$$

where we used $\kappa = 0.34$.

- If the Eddington luminosity starts to approach this a good fraction of this value, then equilibrium is lost and severe mass loss occurs.

3.1.7 Final tools

- There are several manipulations we can carry out to make the expressions we derived more useful for later.
- For future use we will need different forms of Equation (3.8). Take hyrdrostatic equilibrium and use logarithmic derivatives:
$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r} = -\frac{Gm\rho}{rP}. \tag{3.33}$$
- Dividing both sides by $\mathrm{d}\ln T/\mathrm{d}\ln r$ gives a new quantity we'll call "del"
$$\nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = -\frac{r^2 P}{Gm\rho}\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{3.34}$$
which is the true driving gradient in the star.
- If we now consider that the luminosity L is carried ONLY by radiation, then we can define "delrad"
$$\nabla_{\mathrm{rad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{P\kappa_{\mathrm{R}}}{T^4}\frac{L}{m}, \tag{3.35}$$
where we used Equation (3.8).
- So if $\nabla = \nabla_{\mathrm{rad}}$, the all the luminosity is radiative. If $\nabla_{\mathrm{rad}} > \nabla$, there is some other transport mechanism of the energy in addition to radiation.
- This quantity is the local slope which is required if all the luminosity were carried by radiation through diffusion.
- In fact, we will use this as a comparison in this unit to a similar quantity we've already introduced in Equation (2.81),
$$\nabla_{\mathrm{ad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}. \tag{3.36}$$
where this is defined in an "adiabatic" sense, or, i.e., at constant entropy.
- The value of 0.4 comes when considering an ideal gas:
$$\nabla_{\mathrm{ad}} = \frac{\frac{5}{3} - 1}{\frac{5}{3}} = \frac{2}{5}. \tag{3.37}$$

3.2 Conduction

- Degenerate electrons (in white dwarfs or supergiant cores) are the primary carrier for energy transport in such objects.
- The process is again diffusion of the Fick's Law type
$$F_{\mathrm{cond}} = -D_{\mathrm{e}}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{3.38}$$
- The diffusion coefficent D_{e} can again be expressed by an "opacity" of sorts, κ_{cond}, and put into a form similar to Equation (3.7)
$$F_{\mathrm{cond}} = -\frac{4ac}{3}\frac{T^3}{\kappa_{\mathrm{cond}}\rho}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{3.39}$$

- Assume we can compute $\kappa_{\rm cond}$. Then the total energy flux in the star (so far) would be

$$F_{\rm tot} = F_{\rm rad} + F_{\rm cond} = -\frac{4ac}{3}\frac{T^3}{\kappa_{\rm tot}\rho}\frac{{\rm d}T}{{\rm d}r}, \tag{3.40}$$

where

$$\frac{1}{\kappa_{\rm tot}} = \frac{1}{\kappa_{\rm R}} + \frac{1}{\kappa_{\rm cond}}. \tag{3.41}$$

- Note again how opacities are not simply "additive."
- Realize that whatever opacity is smaller is the one that contributes most to the total opacity and thus determines the energy flux (or lack thereof).
- For example, in typical non-degenerate stellar matter, $\kappa_{\rm cond}$ is large (so conduction is negligible), and radiative opacities dominate. Think of it as the one with biggest "channel" that lets the heat through.
- From solid-state physics, one can show that

$$\kappa_{\rm cond} \approx 4\times 10^{-8}\frac{\mu_e^2}{\mu_{\rm I}}Z_c^2\left(\frac{T}{\rho}\right)^2. \tag{3.42}$$

- In a white dwarf, one may encounter $\rho \approx 10^6\,{\rm g\,cm^{-3}}$ and $T \approx 10^7$K, made of carbon.
- The radiative opacity in this environment $\kappa_{\rm R} \approx 0.2\,{\rm cm^2\,g^{-1}}$ (Equation (3.22)). With $\mu_e = 2$, $\mu_{\rm I} = 12$, and $Z_c = 6$, we find $\kappa_{\rm cond} \approx 5\times 10^{-5}\,{\rm cm^2\,g^{-1}}$.
- Thus the total opacity is dominated conduction, and the flux is carried out by conduction.

3.3 Convection

Another important carrier of energy from the stellar interior outward is convection.

3.3.1 The convective instability

- Consider in what follows an ideal gas.
- Assume a blob of gas of density ρ and pressure P at point r. It is in equilibrium with its surroundings also then of density ρ and pressure P.
- Let's displace the blob, or perturb it vertically into the medium (at $r + \delta r$) which now has density ρ' and pressure P', which we know are less than the unprimed quantities. What happens to the blob?
- Let ρ^* be the density of the blob. If $\rho^* < \rho'$ then the blob will be buoyant and continue rising: this is unstable. If $\rho^* > \rho'$ then the blob will return to its original position and there is no instablility. So how do ρ^* and ρ' compare?
- Two physically-motivated assumptions: (1) The pressure imbalances are quickly removed by acoustic waves (on the dynamical time scale), so that the pressure of the blob is also P'. (2) Heat is exchanged on the thermal timescale, which is long, so this is an adiabatic displacement.
- We know for an adiabatic displacement that $P/\rho^\gamma = $ const (Equation (2.79)). Comparing at bottom and top we can show

$$\rho^* = \rho\left(\frac{P'}{P}\right)^{1/\gamma}. \tag{3.43}$$

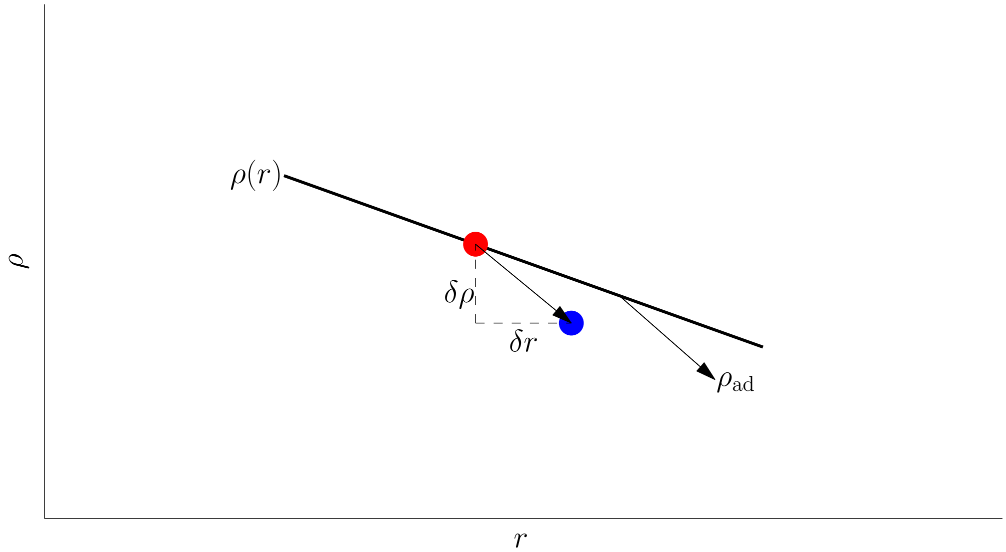

Figure 3.5: Convective instability. The curve $\rho(r)$ denotes the density gradient in some small region of a stellar interior. The arrow is the direction of an adiabat for this material. Take a parcel (red dot) in equilibrium with density ρ, and displace it upwards ($\delta r > 0$) adiabatically. It ends up where the blue dot is. This parcel now has a lower density than the surroundings ($\delta\rho < 0$), and so will continue to rise toward the surface until the conditions change (if they change). The density does not decrease sufficiently fast enough to be stable to convection.

- Let's expand the environmental pressure and density about point r to first order:

$$P' = P(r + \delta r) = P(r) + \frac{\mathrm{d}P}{\mathrm{d}r}\delta r + \ldots \tag{3.44}$$

$$\rho' = \rho(r + \delta r) = \rho(r) + \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r + \ldots \tag{3.45}$$

- Substitute Equations (3.44)-(3.45) into (3.43) and expand (binomial):

$$\rho^* = \rho + \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r. \tag{3.46}$$

- For an instability to occur, $\rho^* - \rho' < 0$, or

$$\rho^* - \rho' = \frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r - \frac{\mathrm{d}\rho}{\mathrm{d}r}\delta r < 0. \tag{3.47}$$

- So, an instability occurs if

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} < \frac{\mathrm{d}\rho}{\mathrm{d}r}, \tag{3.48}$$

  where we introduced the adiabatic gradient

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} = \frac{1}{\Gamma}\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{3.49}$$

  where we've denoted $\gamma = \Gamma$ in the adiabatic case.

- This can be interpreted as the density gradient resulting from adiabatic motion in the given pressure gradient.
- Since the gradient of pressure is always negative (hydrostatic equilibrium), instability occurs when the density does not decrease sufficiently rapidly compared to the adiabatic case.
- See Figure 3.5 for a schematic of this.