October 19 ....

2.6.4 Usefulness of polytropes

- First note that unless K is fixed by a known equation of state (like it is for degenerate gases and a few other cases), then one needs to provide a value of n, M, and R to get physical values for the interior properties.
- A degenerate gas is a case where K is fixed by the equation of state. Recall the non-relativistic case in Equation (2.53):
$$P_\mathrm{e} = 1.0036 \times 10^{13} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3} \ \mathrm{dyne\,cm^{-2}}.$$
- The exponent on density implies $n = 3/2$.
- Using the expression for K in Equation (2.150) along with the necessary computed values gives
$$K = 2.477 \times 10^{14} \left(\frac{M}{M_\odot}\right)^{1/3} \frac{R}{R_\odot}. \tag{2.167}$$
- This yields a mass-radius relationship
$$\frac{M}{M_\odot} \approx 2.08 \times 10^{-6} \left(\frac{2}{\mu_\mathrm{e}}\right)^5 \left(\frac{R}{R_\odot}\right)^{-3}. \tag{2.168}$$
Note how this implies that adding mass decreases the star's radius.
- White dwarfs are measured to have masses around $0.6\,M_\odot$. For completely ionized gas, $\mu_\mathrm{e} = 2$, and this gives a radius of about $R \approx 0.015\,R_\odot$, which is about the radius of the Earth.
- In deriving Equation (2.53) we could have substituted the mass of the neutron instead of the electron. Neutrons can become degenerate in special cases too. Then taking $\mu_\mathrm{e} = 1$, the *neutron star* equivalent of Equation (2.168) is
$$\frac{M}{M_\odot} \approx 5 \times 10^{-15} \left(\frac{R}{R_\odot}\right)^{-3}. \tag{2.169}$$
- For a $1\,M_\odot$ star, we find $R \approx 10\,\mathrm{km}$.
- For relativistic electrons, we had from Equation (2.56)
$$P_\mathrm{e} = 1.243 \times 10^{15} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3} \ \mathrm{dyne\,cm^{-2}}.$$
- This now implies a polytropic index $n = 3$. Using the proper values for this case gives
$$K = 3.841 \times 10^{14} \left(\frac{M}{M_\odot}\right)^{2/3}, \tag{2.170}$$
or, the relation (independent of radius)
$$\frac{M}{M_\odot} = 1.456 \left(\frac{2}{\mu_\mathrm{e}}\right)^2. \tag{2.171}$$
- This is the Chandrasekhar limiting mass for a degenerate, relativistic electron gas. We'll come back to this again discussing late-stage evolution.

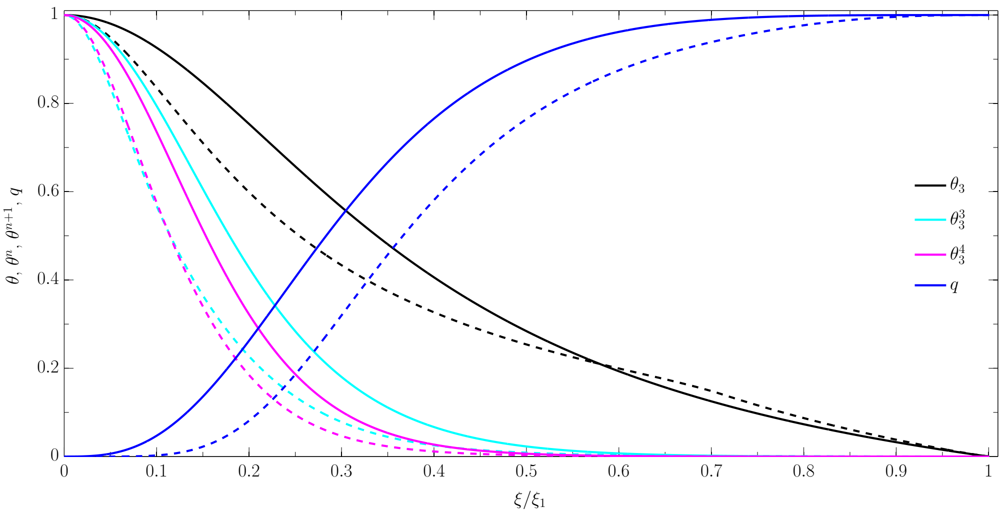

Figure 2.9: Various quantities for a $n = 3$ polytrope compared to Model S. For each colored curve the solid line is the polytropic solution, and the dashed line is the model Sun.

- Another interesting insight is for radiation pressure.
- For an ideal gas, recall Equation (2.83)

$$P = \frac{R}{\mu} k_{\rm B} T + \frac{1}{3} a T^4 = \frac{R}{\mu\beta} \rho T,$$

  where $\beta = P_{\rm gas}/P$ is assumed constant throughout the star.
- Since $1 - \beta = P_{\rm rad}/P$, this shows that $T^4 \sim P$.
- Using this to solve for P, we can find

$$P = \left(\frac{3R^4}{a\mu^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3}, \tag{2.172}$$

  which demonstrates this is an $n = 3$ polytrope.
- Here, K is again a free parameter given some value of β. However $n = 3$ polytropes are a bit delicate.
- What in practice is necessary, is to choose a mass M, and through Equation (2.150) compute the value of $(1 - \beta)/\mu\beta^4$.
- However, for a given mass, there is an infinite number of models for various R. Nonetheless, for large supermassive models, one sees the radiation pressure dominating the equation of state. We'll come back to this in Sec. 3.1.6.
- Finally, consider the Sun.
- It is apparent from the polytrope solutions (Table 2.1) that Θ_n does not change much with n, so our approximation for the central temperature found in Problem 2.6 is quite good, where we effectively found that $\Theta = 1$. The numerical central temperature is $T_c = 1.57 \times 10^7$ K.
- We see that other solutions however vary quite widely for different indices. For example, The W_n can be directly compared with our prior estimates of central pressure in Section 2.4.1. For the Sun we found that the constant is about $261/4\pi \approx 21$, with total numerical value $P_c = 2.34 \times 10^{17}\,\mathrm{dyne\,cm^2}$. This is the first piece of evidence that the Sun is well described by a $n = 3$ polytrope.

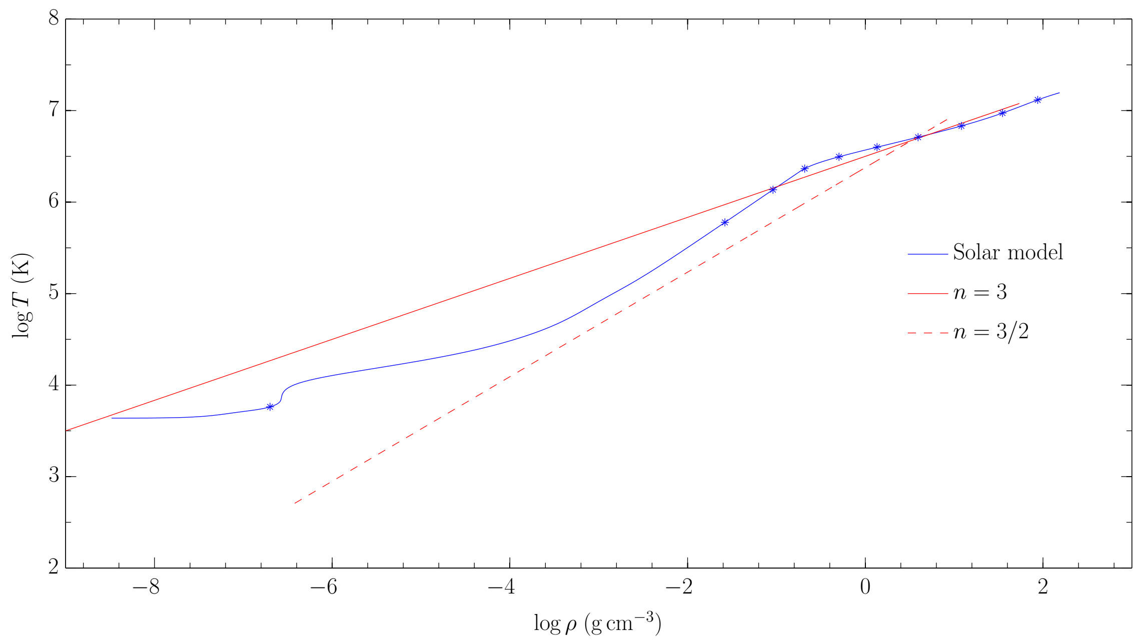

Figure 2.10: A modern solar model compared to 2 different polytropic models. The blue dots on the solar model denote the interior radial coordinate from $r = (0.1 - 1.0)R_\odot$ in steps of 0.1. This particular solar model goes slightly beyond the photosphere.

- We also know the Sun's central density is about $\rho_c = 153\,\mathrm{g\,cm^{-3}}$, and ratio with mean density $153/1.4 \approx 110$.
- All of these comparisons point to the Sun being somewhere between $n = 3.0 - 3.5$, at least in the central regions.
- In Figure 2.9 we plot a $n = 3$ polytrope and compare with a modern solar model (Model S). What is remarkable is how well the polytrope approximates the Sun without any knowledge of energy generation or energy transfer or chemical composition variations.
- Figure 2.10 shows a detailed solar model in the $T - \rho$ landscape, compared to an $n = 3$ and $n = 3/2$ polytrope. Near the core, the polytrope is in good agreement with the Sun since the density is quite large. Recall an $n = 3$ polytrope corresponds to $\gamma = 1 + 1/3 = 4/3$, which is the degenerate electron gas case.
- However in the outer layers, the Sun behaves more like an ideal gas because of convection, so a $\gamma = 5/3$, $n = 1/(\gamma - 1) = 1.5$ polytrope is a better approximation.
- Note that the layers from $r = 0.7$ outward constitutes 30% of the star in radius, but only about 0.5% in mass.