- Note that it is convention to express Equation (3.48) as

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P} < \frac{1}{\Gamma_1}. \tag{3.50}$$

(Note since we've divided by a negative number, $\mathrm{d}\ln P/\mathrm{d}r$, the inequality changes). For a fully ionized ideal gas, the RHS is 3/5.

- Let's now consider the force per unit volume acting on the displaced blob. That force (buoyancy and gravitational) is $F = -(\rho^* - \rho')g$, since g acts downwards.

IN CLASS WORK

Use this force in Newton's second law and derive a simple equation of motion for the displacement δr. Show that a characterstic frequency N comes out

$$N^2 = \frac{g}{\rho*}\left(\frac{\rho}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = \frac{g}{\rho*}\left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\mathrm{ad}} - \frac{\mathrm{d}\rho}{\mathrm{d}r}\right], \tag{3.51}$$

called the Brunt-Väisälä frequency. Examine the solutions of the equation of motion based on the possible values of N in the stable or unstable condition.

Answer:
From Newton's second law

$$\rho*\frac{\mathrm{d}^2\delta r}{\mathrm{d}t^2} = -(\rho^* - \rho')g.$$

If we plug in Equation (3.47) we get the equation of motion

$$\frac{\mathrm{d}^2\delta r}{\mathrm{d}t^2} + N^2\delta r = 0,$$

where N is the Brunt-Väisälä frequency given above.
A general solution to this equation is $\delta r \propto \mathrm{e}^{\pm iNt}$.
If the medium is stable to convection, we know that $N^2 > 0$. When this is the case, the solution is thus sinusoidal and the blob δr oscillates about a given point (gravity/buoyancy waves).
In the other case, $N^2 < 0$ and so N is imaginary: $N \to iN$. The the solution goes as $\delta r \propto \mathrm{e}^{-Nt} + \mathrm{e}^{Nt}$. This solution describes an exponentially growing parcel, in other words, a convective instability.

- It is convenient to look at the stability criterion in terms of temperature gradients instead.

- Using Eq. (3.45) and the ideal gas law $P = \rho RT/\mu$ (ignore gradients in mean molecular weights FOR NOW) we can show that

$$\rho' = \rho + \frac{\rho}{p}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r - \frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\delta r. \tag{3.52}$$

- For instability, again, we then require that $\delta\rho < 0$, or

$$\rho^* - \rho' = \left(\frac{1}{\gamma} - 1\right)\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\delta r + \frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\delta r < 0. \tag{3.53}$$

- Now the instability condition becomes

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} > \frac{\mathrm{d}T}{\mathrm{d}r}, \tag{3.54}$$

where the adiabatic temperature gradient is given by

$$\left(\frac{dT}{dr}\right)_{ad} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}. \tag{3.55}$$

- This says that if the temperature gradient decreases too steeply out through the star there will be convection.

- To simplify in analogy with what we did before (Eq. 3.50), it is convention to write the inequality as

$$\frac{d\ln T}{d\ln P} > \frac{\Gamma_2 - 1}{\Gamma_2}. \tag{3.56}$$

- We reintroduce

$$\nabla = \frac{d\ln T}{d\ln P}, \qquad \nabla_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2}, \tag{3.57}$$

so that the instability condition is

$$\nabla > \nabla_{ad}. \tag{3.58}$$

Again, the RHS is 2/5 in the ionized ideal case.

- This is known as the **Schwarzschild criterion**. The inequality is also referred to as superadiabaticity.

3.3.2 Another useful formulation

- Convection can also be understood in terms of entropy.

- For reversible processes, $dQ = TdS$ and so

$$T\,dS = dU + PdV. \tag{3.59}$$

- Using standard thermodynamic relations, it can be shown from here that

$$\frac{dS}{dr} = c_P(\nabla - \nabla_{ad})\frac{d\ln P}{dr}. \tag{3.60}$$

- So if the star is radiative, $dS/dr > 0$ and the entropy increases outward.

- If the star is convective, $dS/dr < 0$. If the convection is efficient the gradient is very close to being adiabatic, meaning the entropy is very nearly constant throughout convection zones.

3.3.3 Semiconvection

- Let's return to the Brunt-Väisälä frequency again from Equation (3.51):

$$N^2 = g\left(\frac{1}{\gamma P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right).$$

- We want to rewrite this in a very convenient form, and this time we **will** take into account composition gradients in the gas to be as general as possible. The form is

$$N^2 = \frac{g^2\rho}{P}(\nabla_{ad} - \nabla + \nabla_\mu), \tag{3.61}$$

where (as some have been defined before)

$$\nabla = \frac{d\ln T}{d\ln P}, \qquad \nabla_{ad} = \left(\frac{d\ln T}{d\ln P}\right)_{ad}, \qquad \nabla_\mu = \frac{d\ln \mu}{d\ln P}. \tag{3.62}$$

PROBLEM 3.3:   [5 pts]: Show how you can get from Equation (3.51) to Equation (3.61). Start with

$$N^2 = g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right),$$

$$P = \frac{\rho T}{\mu},$$

$$\mathrm{d}\ln P = \mathrm{d}\ln\rho + \mathrm{d}\ln T - \mathrm{d}\ln\mu.$$

- Ignore the composition gradient for a second. We recover the "standard" stability relation: if the temperature gradient is larger than the adiabatic one (Schwarzchild), the BV frequency becomes complex.

- If it's the reverse, the BV is positive and the medium is stable to convection.

- Now however, we have the possibility that the Schwarzchild criterion is satisfied ($\nabla > \nabla_{\mathrm{ad}}$), yet the medium remains stable because the composition gradient makes it positive again.

- This is the *Ledoux criterion*, and when this is the case we have weak convection, or **semiconvection**.

- This typically would not occur in a convection zone, why? Because convection mixes material and composition gradients are removed.

- But in areas of nuclear burning where gradients do exist, and at the "edges" of convection zones, this situation can arise. Large peaks in the μ-gradient (also caused by g increases) can cause large jumps in N.

- Can be thought of as difficulty in moving "heavier" material (high μ) up - it doesn't want to do that.

- Becomes important for red-giant stars and their gravity modes mixing (boosting frequency) with acoustic modes.

3.3.4   One more useful formulation

- Start with Equation (3.55) and rewrite using ideal gas law and hydrostatic equilibrium:

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = \left(1 - \frac{1}{\gamma}\right)\frac{\mu}{\rho R_g}\frac{\mathrm{d}P}{\mathrm{d}r} = -\left(1 - \frac{1}{\gamma}\right)\frac{g\mu}{R_g}. \tag{3.63}$$

- Remembering that $\gamma = c_P/c_V$ and $c_P - c_V = R_g/\mu$,

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = -\left(\frac{c_P/c_V - 1}{c_P/c_V}\right)\frac{g\mu}{R_g}, \tag{3.64}$$

$$= -\left(\frac{\frac{c_P - c_V}{c_V}}{c_P/c_V}\right)\frac{g\mu}{R_g}, \tag{3.65}$$

$$= -\left(\frac{c_P - c_V}{c_P}\right)\frac{g\mu}{R_g}, \tag{3.66}$$

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}} = -\frac{g}{c_P}. \tag{3.67}$$

- This form shows how the parcel of gas is changing as it rises adiabatically and expands

- This can also be derived using energy by considering energy release and work against gravity