Unit 4

The Main Sequence

4.1 Summary of stellar structure

- Mass and radius relationship

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho$$

- Hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho$$

- Energy

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \varepsilon$$

- Energy transport

$$L = 4\pi r^2 (F_{\mathrm{rad}} + F_{\mathrm{conv}})$$

  - Radiation

$$F_{\mathrm{rad}} = -\frac{4acT^3}{3\kappa_{\mathrm{R}}\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$

  - Convection

$$F_{\mathrm{conv}} = \rho C_p T (g\delta)^{1/2}\frac{\ell^2}{4}H_p^{-3/2}(\nabla - \nabla_{\mathrm{ad}})^{3/2}.$$

- Equation of state

$$P = \frac{\rho k_{\mathrm{B}} T}{\mu m_{\mathrm{u}}} + \frac{a}{3}T^4$$

- Rosseland opacity

$$\kappa_{\mathrm{R}} = \kappa_{\mathrm{R}}(\rho, T, \mu)$$

- Energy generation

$$\varepsilon = \varepsilon(\rho, T, \mu) = \varepsilon_{\mathrm{nuc}} + \varepsilon_{\mathrm{grav}}$$

  where

$$\varepsilon_{\mathrm{grav}} = -T\frac{\mathrm{d}S}{\mathrm{d}t}$$

  is related to Kelvin-Helmholtz contraction (and note the time dependence).

- Abundance changes

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{\varepsilon_{\mathrm{nuc}}}{26.7\,\mathrm{MeV}}$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -\frac{\mathrm{d}X}{\mathrm{d}t}$$

- Boundary conditions

$$
\begin{aligned}
r &\longrightarrow 0; & m(r) &\longrightarrow 0 \\
r &\longrightarrow 0; & L(r) &\longrightarrow 0 \\
r &\longrightarrow R; & m(r) &\longrightarrow M \\
r &\longrightarrow R; & \rho(r) &\longrightarrow 0 \\
r &\longrightarrow R; & T(r) &\longrightarrow T_{\mathrm{eff}}
\end{aligned}
$$

- Vogt-Russell Theorem

  - There are only 2 free parameters in the equations needing to be solved: the total stellar mass M and the chemical composition.

  - The theorem states that just these 2 parameters uniquely determine the structure. Evolution is only based on the changing of the composition (mainly) and mass (sometimes) due to nuclear burning.

4.2   Homology relations for stars in radiative equilibrium

4.2.1   Basic idea

- Solving the equations of stellar structure is possible, yet difficult.

- There are ways of obtaining useful insights without doing so (like using the first 2 equations to study polytropes).

- Consider we are dealing with stars that are homologous: that a star with mass, say, M is a scaled version of a star of mass M' (if mass doesn't change too rapidly).

- Homologous points on homologous mass shells means that $m/M = m'/M'$.

- This assumption allows us to find relations between one numerical solution and another, without computing more than one solution.

- We will use an ideal gas equation of state.

- One way of doing this is to introduce dimensionless variables

$$\tilde{r} = \frac{r}{R_0}; \quad \tilde{m} = \frac{m}{M_0}; \quad \tilde{L} = \frac{L}{L_0}; \quad \tilde{T} = \frac{T}{T_0}; \quad \tilde{P} = \frac{P}{P_0}. \tag{4.1}$$

- T_0 and P_0 are chosen in such a way so that the basic structure equations are simplified

$$T_0 = \frac{\mu G M_0}{R_0 R_{\mathrm{g}}}; \qquad P_0 = \frac{G M_0^2}{4\pi R_0^4} \tag{4.2}$$

- The 0 subscripts can be thought of as a reference star.

- Then, hydrostatic equilibrium becomes

$$\frac{d\tilde{P}}{d\tilde{r}} = -\frac{\tilde{P}}{\tilde{T}}\frac{\tilde{m}}{\tilde{r}^2}. \tag{4.3}$$

- The mass-radius equation is

$$\frac{d\tilde{m}}{d\tilde{r}} = \frac{\tilde{P}}{\tilde{T}}\tilde{r}^2. \tag{4.4}$$

- The temperature gradient for radiative equilibrium is

$$\frac{d\tilde{T}}{d\tilde{r}} = -C\frac{\tilde{L}}{\tilde{r}^2}\frac{\tilde{P}^{\alpha+1}}{\tilde{T}^{\alpha+\beta+1}}, \tag{4.5}$$

if an opacity is considered of the form $\kappa_R = \kappa_0 \rho^\alpha T^{-\beta}$.

- The energy equation is

$$\frac{d\tilde{L}}{d\tilde{r}} = D\tilde{P}^2\tilde{T}^{\nu-2}\tilde{r}^2 \tag{4.6}$$

if the energy is considered to be of the form $\varepsilon = \varepsilon_0 \rho T^\nu$.

- The constants that have appeared are

$$C = C_0\frac{\kappa_0}{\mu^{\beta+4}}\frac{L_0 R_0^{\beta-3\alpha}}{M_0^{\beta+3-\alpha}}, \tag{4.7}$$

$$C_0 = \frac{3}{16\sigma}\left(\frac{R_g}{G}\right)^{\beta+4}\left(\frac{1}{4\pi}\right)^{\alpha+2}, \tag{4.8}$$

$$D = D_0\varepsilon_0\mu^\nu\frac{M_0^{\nu+2}}{L_0 R_0^{\nu+3}}, \tag{4.9}$$

$$D_0 = \left(\frac{G}{R_g}\right)^\nu\frac{1}{4\pi}. \tag{4.10}$$

- These are the equivalent *dimensionless* equations of stellar structure, along with accompanying boundary conditions.

- C_0 and D_0 are just fundamental constants.

- Note that C and D depend only on M_0, R_0, and L_0.

- The two center boundary conditions, $\tilde{m} = \tilde{L} = 0$ at $\tilde{r} = 0$, imply that there is only one pair of constants C and D that satisfies these conditions.

- Thus, there is only one solution for stars in radiative equilibrium! All those stars have the same C and D.

- All stars are not radiative, there are convection zones, and this messes things up a bit, which will be shown later.

- Therefore, these ideas work well for stars with small convection zones, such as most A, F, G, and some B stars (where we also ignore degeneracy and radiation pressure).

- In what follows, we can study 4 cases: the CNO cycle ($\nu \simeq 20$) and the PP-chain ($\nu \simeq 5$) stars, as well as Kramer's opacities ($\beta = 3.5$, $\alpha = 1$) or electron scattering ($\beta = \alpha = 0$) cases.

4.2.2   Dependence on mass

- Drop the zero subscripts for now.

- Here we look at $R(M)$, $T(M)$, and $L(M)$.

- Note that the produce of constants CD is independent of luminosity (ignore composition for a moment, assume all stars are the same)

$$CD = \text{const } \frac{R^{\beta-3\alpha}M^{\nu+2}}{R^{\nu+3}M^{\beta-\alpha+3}}. \tag{4.11}$$

- So we have

$$R^{\nu+3-\beta+3\alpha} \propto M^{\nu-1-\beta+\alpha}. \tag{4.12}$$

	$\nu = 5$	$\nu = 20$
e-scattering	$R \propto M^{0.5}$	$R \propto M^{0.83}$
Kramers	$R \propto M^{0.2}$	$R \propto M^{0.73}$

- Just to note, as we saw earlier, degenerate stars shrink with increasing mass, since these are polytropes with $n = 3$ and the exponent is $-1/3$.

- From the ideal gas law and hydrostatic equilibrium the central temperature $T_c \propto M/R$.

- This can be used along with Equation (4.12) to find the central temperature dependence on mass alone

	$\nu = 5$	$\nu = 20$
e-scattering	$T_c \propto M^{0.5}$	$T_c \propto M^{0.17}$
Kramers	$T_c \propto M^{0.8}$	$T_c \propto M^{0.27}$

- So clearly with increasing mass, the central temperature must increase.

- Thus we must go from PP chain to CNO burning in interiors. We can't say anything about effective temperature yet because the photosphere has different opacity laws.

- Finally C can give us the relation for luminosity

$$L \propto \frac{M^{\beta+3-\alpha}}{R^{\beta-3\alpha}} \tag{4.13}$$

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto M^{3.0}$	$L \propto M^{3.0}$
Kramers	$L \propto M^{5.4}$	$L \propto M^{5.14}$

- Note that for electron scattering the luminosity does not depend on the energy generation "type."

- For the CNO cycle values, this best matches A and F stars.

- Note that the mass-luminosity relations follow mostly from C, which describes energy transport (from the $\mathrm{d}T/\mathrm{d}r$ equation).

- Briefly, note from the constant C that we can get an expression for the luminosity in scaled solar values considering a Kramer's opacity:

$$L \simeq 1.4 \times 10^{35} \left(\frac{M}{M_\odot}\right)^{5.5} \frac{1.7}{1+X}\frac{0.02}{Z}\left(\frac{\mu}{0.62}\right)^{7.5}\left(\frac{R}{R_\odot}\right)^{-0.5} \text{erg s}^{-1}. \tag{4.14}$$

- The solar luminosity is about $3.9 \times 10^{33} \text{ erg s}^{-1}$, so this is a bit high, but still impressive.

- What about T_{eff}? Since $T_{\text{eff}}^4 \propto L/R^2$, this is easy.

	$\nu = 5$	$\nu = 20$
e-scattering	$T_{\rm eff} \propto M^{0.5}$	$T_{\rm eff} \propto M^{0.34}$
Kramers	$T_{\rm eff} \propto M^{1.25}$	$T_{\rm eff} \propto M^{0.92}$

4.2.3   Dependence on $T_{\rm eff}$

- For a proper H-R diagram we'd like to have $L \propto T_{\rm eff}^{\gamma}$. We can do that with what we've just found since we have $L \propto R^2 T_{\rm eff}^4$ and we have both of those as relations to the mass.

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto T_{\rm eff}^{6.0}$	$L \propto T_{\rm eff}^{8.9}$
Kramers	$L \propto T_{\rm eff}^{4.32}$	$L \propto T_{\rm eff}^{5.6}$

4.2.4   Dependence on mean molecular weight

- We ignore here He abundance which typically enters through κ_0, which contains terms with the number of electrons, etc.

- We multiply C and D but keep the μ dependence now

$$CD = \text{const} \, \frac{\mu^{\nu}}{\mu^{\beta+4}} \frac{R^{\beta-3\alpha} M^{\nu+2}}{R^{\nu+3} M^{\beta-\alpha+3}}. \tag{4.15}$$

- The idea is first to compare stars of the same mass and see how things change with μ.

- The radius can go either way!

	$\nu = 5$	$\nu = 20$
e-scattering	$R \propto \mu^{1/8}$	$R \propto \mu^{0.7}$
Kramers	$R \propto \mu^{-1/3}$	$R \propto \mu^{0.6}$

- What about luminosity? We can use constant C again

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto \mu^{4.0}$	$L \propto \mu^{4.0}$
Kramers	$L \propto \mu^{7.67}$	$L \propto \mu^{7.2}$

- We see that the luminosity increases steeply with increasing μ, more than with mass!

- A higher central temperature is required to increase the central pressure to balance the heavier material, hence higher luminosity. This will be a critical point for understanding different phases of stellar evolution.

- The $T_{\rm eff}$ effect on molecular weight is not insignificant

	$\nu = 5$	$\nu = 20$
e-scattering	$T_{\rm eff} \propto \mu^{0.94}$	$T_{\rm eff} \propto \mu^{0.65}$
Kramers	$T_{\rm eff} \propto \mu^{2.1}$	$T_{\rm eff} \propto \mu^{1.5}$

- Stars move up and to the left of the HR diagram as the mean molecular weight increases.

- At the same time, stars have a less-steep HR curve for the same M and different μ than for the same μ and different M.
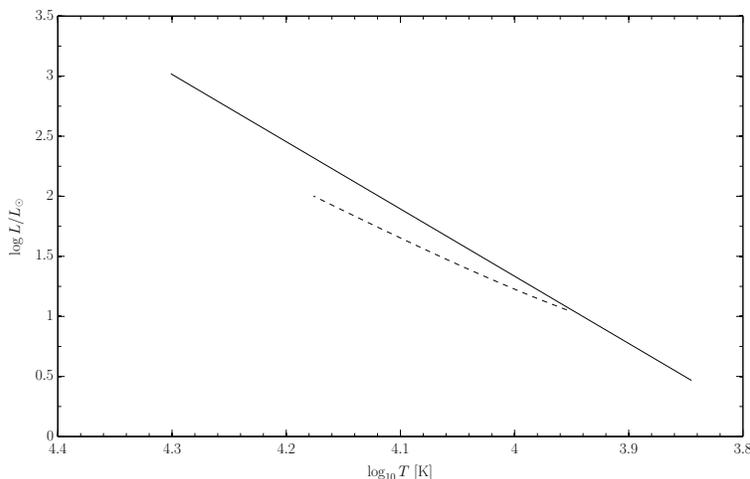
Figure 4.1: The solid line represents the main sequence for stars of various mass but fixed μ in the CNO cycle, so $L \propto M^{5.6}$. The dashed line is the evolutionary track for a star with a given mass but increasing μ, so $L \propto \mu^{4.8}$.

	$\nu = 5$	$\nu = 20$
e-scattering	$L \propto T_{\text{eff}}^{4.25}$	$L \propto T_{\text{eff}}^{6.2}$
Kramers	$L \propto T_{\text{eff}}^{3.65}$	$L \propto T_{\text{eff}}^{4.8}$

- So, we see that stars are hotter and more luminous with increasing μ. The question is are they above or below the main sequence of H-rich stars?

- Firstly, the luminosity increases more slowly for He stars than H stars for a given temperature.

- Secondly, for increasing μ, T_{eff} increases more steeply with L than for H-rich stars.

- Therefore, it turns out that He-rich stars, for a given mass, have a higher L and a higher T_{eff} than a "solar" main sequence, and thus falls "below" it. See Figure 4.1.

4.2.5   Dependence on heavy metal abundances

- Let's look at lower-mass stars and their dependence on the metal mass fraction Z.

- This is only approximate because the depth of the outer convection zone depends on Z, but for homologous stars this will still give the right trend.

- The metals appear mainly through the κ_0 term in the bound-free processes in deep interiors where only metals still have electrons.

- Furthermore there is no dependence of ε_0 on Z in the PP-chain reactions.

- So considering $\beta = 3.5$ and $\alpha = 1$ we have for the product CD

$$CD = \text{const}\, \frac{Z R^{0.5} M^{\nu+2}}{R^{\nu+3} M^{5.5}}. \tag{4.16}$$

Kramers, $\nu = 5$	$R \propto Z^{0.13}$	$L \propto Z^{-1.1}$	$T_{\text{eff}} \propto Z^{-0.34}$	$L \propto T_{\text{eff}}^{3.24}$

- For stars of a given mass the luminosity increases with decreasing metals.

- The lifetime of a metal-poor star is shorter than for a metal-rich star: it burns through its fuel faster.

- The opacity decreases with smaller Z and radiation escapes more easily.

- For stars with the same mass but different Z, $L \propto T_{\text{eff}}^{3.24}$, which is less steep than stars for the same Z but different mass, $L \propto T_{\text{eff}}^{4.3}$, and less steep than one for changing mean molecular weight too.

- Stars with few metals, like Population II ones, would have a main sequence **below** the solar abundance one (for low mass).

4.2.6 Contracting stars in radiative equilibrium

- Consider a star contracting from gravitational energy release in its young life.

- Its temperature is not high enough for nuclear reactions to take place, so the quantity D is not applicable here.

- But we can use C. During contraction, the mass stays relatively constant, as does composition and μ.

- For Kramer's we find

Kramers	$L \propto R^{-0.5}$	$T_{\text{eff}} \propto R^{-0.63}$	$T_{\text{eff}} \propto L^{1.1}$	$L \propto T_{\text{eff}}^{0.95}$

- But is this what we observe? Kind of for high-mass stars. As they contract they slightly increase in luminosity.

- And the temperature increases across the HR diagram

- But not at all for low-mass stars, where they contract at constant temperature along almost vertical lines.

- Why? Convection needs to be taken into account in low-mass stars.

4.2.7 Convective stars

- Convection zones in stars change them in 2 main ways

  - The radius becomes smaller.
  - Energy transport becomes more effective than radiation (because of the large absorption coefficients) and the temperature gradient gets reduced.

- As energy transport is increased, more energy is lost, gas pressure decreases and gravity starts to dominate ...

- The star contracts, thus increasing the internal temperature and the energy generation ..

- This balances the energy loss and thermal equilibrium is restored, albeit at a smaller radius ...

- The luminosity is larger, thus the effective temperature must be larger ...

- The star moves up and to the left on the HR diagram

- We can treat convective stars similarly with just a change to the equation of state.

- In convection zones

$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln \rho} = \Gamma_1.$$

- We assume $\Gamma_1 = \gamma = 5/3$ is constant .
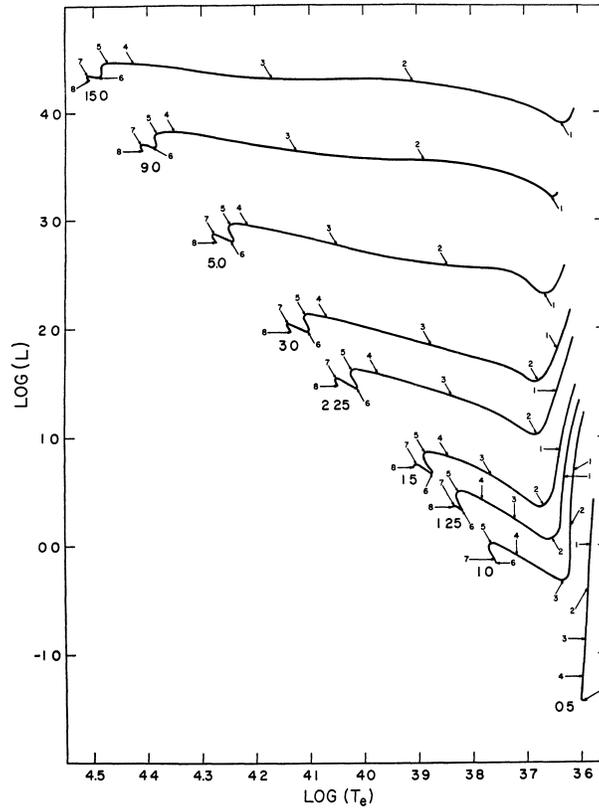
Figure 4.2: Evolutionary tracks for pre-main sequence stellar models of the given masses. Composition is $X = 0.708$ and $Z = 0.02$. From ?.

- Thus we have the polytropic relation we found before by integrating the above equation

$$P = K\rho^{\gamma}.$$

- We can assume for simplicity that a convection zone extends all the way to the surface, and we match to the photosphere by

$$K = \frac{P_{\text{ph}}}{\rho_{\text{ph}}^{\gamma}}. \tag{4.17}$$

- Using radiative transfer at the photosphere (which we have NOT done), one can show that the pressure and density in the above equation give

$$K = \left[\frac{GM(\alpha+1)}{R^2 \kappa_0^{\text{ph}}}\right]^{-\eta} \left(\frac{k_{\text{B}}}{\mu m_{\text{u}}}\right)^{1+\eta} T_{\text{eff}}^{1+\eta(-\beta+1)}, \tag{4.18}$$

where $\eta = (\gamma - 1)/(\alpha + 1)$ and α and β have their usual definitions in the opacity expressions.

- There are a few questionable assumptions so far

    - We assume that convection extends all the way to the surface, but it becomes very inefficient there because of the density fall off.

    - We also assume γ remains constant in the convection zone so that the polytrope relation holds, but in ionization zones it can change quite a bit.

- Anyway, we continue with a polytrope $n = 1.5$ where from before

$$K = K_{\text{poly}} = N_{3/2} G R M^{1/3}. \tag{4.19}$$

- Matching these two Ks and solving for the effective temperature gives

$$T_{\text{eff}} = N_{3/2}^\xi \left(\frac{G\mu}{k_{\text{B}}} \right)^{\xi(\eta+1)} \left(\frac{\kappa_0^{\text{ph}}}{\alpha+1} \right)^{-\xi\eta} R^{\xi(1-2\eta)} M^{\xi(\eta+1/3)}, \tag{4.20}$$

where $\xi = 1/[1 + \eta(\beta + 1)]$.

- Solving for the radius R and using $L = 4\pi\sigma R^2 T_{\text{eff}}^4$ gives an expression for the luminosity, as well as other similar-type relations we've computed before. The results are

$$\begin{aligned}
T_{\text{eff}} &\simeq 2400\,\text{K} \left(\frac{M}{M_\odot} \right)^{0.2} \left(\frac{R}{R_\odot} \right)^{0.06}, \\
\frac{L}{L_\odot} &\simeq 0.03 \left(\frac{M}{M_\odot} \right)^{0.8} \left(\frac{R}{R_\odot} \right)^{2.2}, \\
\frac{L}{L_\odot} &\simeq 0.03 \left(\frac{M}{M_\odot} \right)^{-7.0} \left(\frac{T_{\text{eff}}}{2400\,\text{K}} \right)^{40.0}.
\end{aligned}$$

- Note that

  - The dependence of the temperature on parameters is very weak: these stars all have similar temperatures.
  - This describes the contraction of convective stars as they form.
  - This is known as the *Hayashi track*.
  - The slope of the luminosity (the sign) is wrong a bit, but it's pretty good.
  - There cannot be stars to the right of the Hayashi track (with a lower effective temperature).

- A glance at the approach to the main sequence along the Hayashi track is shown in Figure 4.2.